1

TRANSFORMATIONS

- Translations and dilations
- Combining transformations
- Curve Sketching
- Using graphs to solve equations and inequalities

Exercise 1D

Using graphs to solve equations and inequalities

Fundamentals

Fundamentals 1

The solutions to $f(x) > 0$ can be found by sketching $y =$ _____ and observing the x-coordinates where the curve is a_____ the ___-axis.

Fundamentals 2

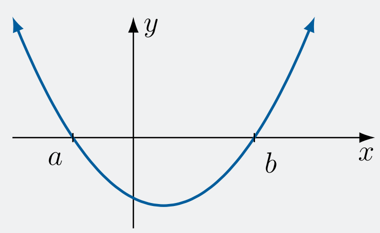
(a) To solve a quadratic inequality in the form $ax^2 + bx + c \geq 0$ or $ax^2 + bx + c \leq 0$, first sketch the graph of $y =$ ________.

(b) Then, depending on the direction of the inequality, shade the region that is either a_____ or b_____ the ___-axis.

(c) The set of ___-values that are shaded is the solution set.

Fundamentals 3

The diagram below shows the sketch of $y = (x - a)(x - b)$.



Write down the inequality that corresponds to

(a) $(x - a)(x - b) \geq 0$

(b) $(x - a)(x - b) < 0$

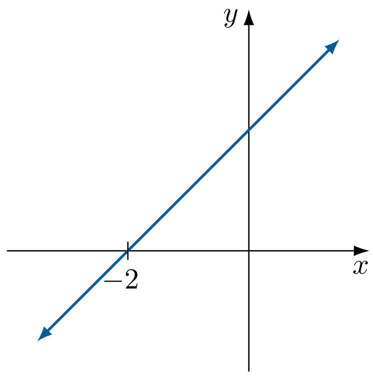
Fundamentals 4

(a) To find the intersection points of $y = f(x)$ and $y = g(x)$, we solve the two equations s________.

(b) The solutions of $f(x) = g(x)$ correspond to the ___-coordinates of where the two graphs intersect.

(c) Hence, the number of solutions to $f(x) - g(x) = 0$ can be found by instead sketching $y =$ _____ and $y =$ _____, then counting how many times they intersect.

Question 1 The diagram below shows the graph of $y = x+2$. Use your graph to solve the following.



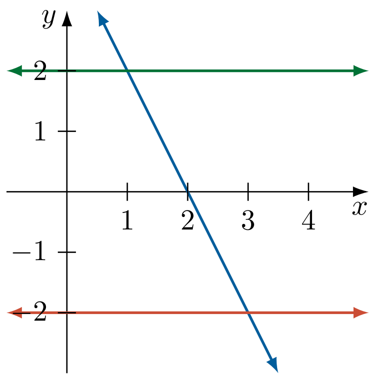
(a) $x + 2 = 0$ (b) $x + 2 < 0$ (c) $x + 2 > 0$

Question 2 By drawing a sketch, or otherwise, solve the following inequalities.

(a) $2x - 4 \leq 0$ (b) $1 - 3x \geq 0$

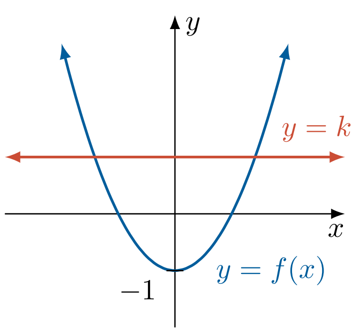
(c) $2 - \frac{1}{2}x < 0$ (d) $\frac{2x}{3} + \frac{1}{2} > 0$

Question 3 The graphs of $y = -2x + 4$ and $y = \pm 2$ are drawn below.



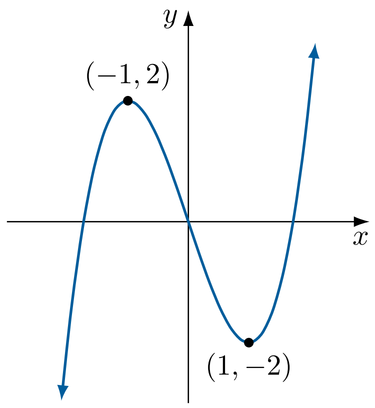
Use this diagram to write down the solution of $-2 \leq 4 - 2x \leq 2$.

Question 4 The diagram below shows the graph of a function $y = f(x)$, and a horizontal line $y = k$. Find the value(s) of k such that $f(x) = k$ has



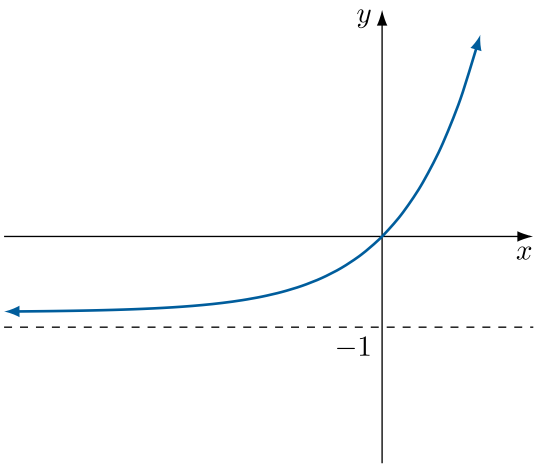
(a) one solution. (b) two solutions. (c) no solutions.

Question 5 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



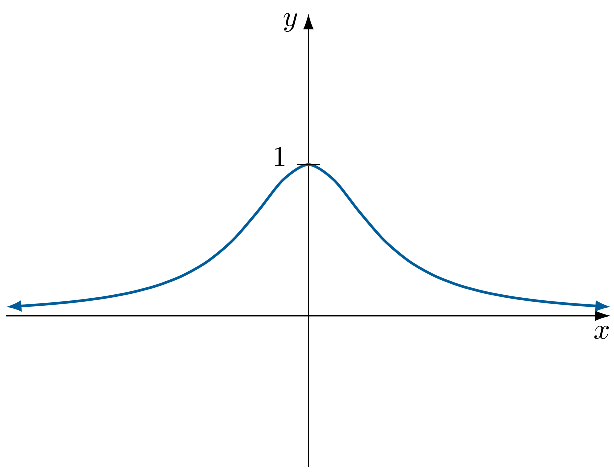
(a) one solution. (b) two solutions. (c) three solutions.

Question 6 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



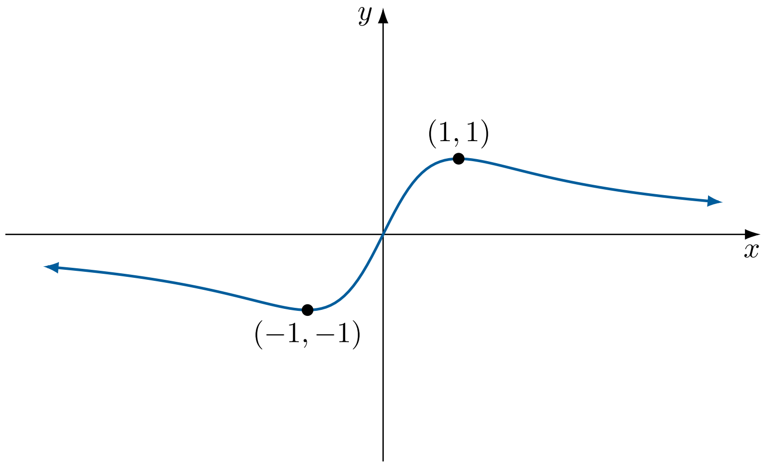
(a) one solution. (b) two solutions. (c) no solutions.

Question 7 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



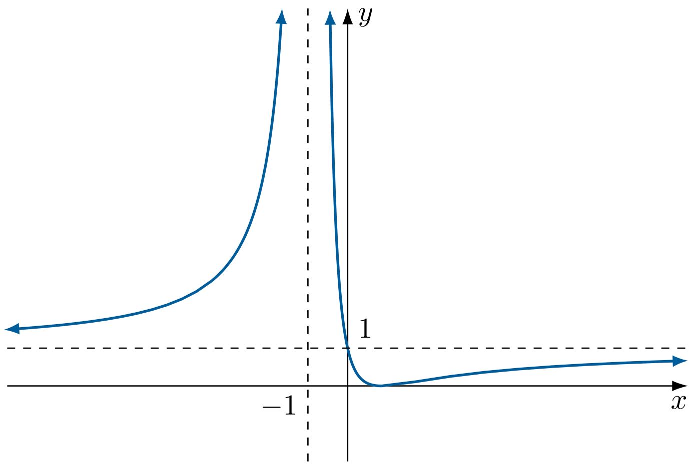
(a) one solution. (b) two solutions. (c) no solutions.

Question 8 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



(a) one solution. (b) two solutions. (c) no solutions.

Question 9 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



(a) one solution. (b) two solutions. (c) no solutions.

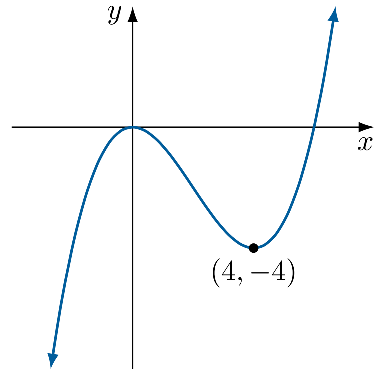
Question 10

(a) Sketch $y = x^2 - 4x + 5$, labelling the vertex.

(b) Use your diagram to find the value(s) of k such that $x^2 - 4x + 5 - k = 0$ has two solutions.

(c) Verify your answer by finding the discriminant of $x^2 - 4x + (5 - k) = 0$.

Question 11 The graph of $y = f(x)$ is sketched below.



(a) For what values of k does the equation $f(x) = k$ have
(i) 1 solution? (ii) 2 solutions? (iii) 3 solutions?

(b) For what values of k does the equation $2f(x) = k$ have
(i) 1 solution? (ii) 2 solutions? (iii) 3 solutions?

(c) For what values of k does the equation $f(x+1) = k$ have
(i) 1 solution? (ii) 2 solutions? (iii) 3 solutions?

Question 12

(a) Sketch the graph of $y = (x-2)(x+1)$, labelling the x-intercepts. You do not need to find the coordinates of the vertex.

(b) Hence, state the value(s) of x for which
(i) $(x-2)(x+1) = 0$. (ii) $(x-2)(x+1) < 0$. (iii) $(x-2)(x+1) > 0$.

Question 13 Use a similar technique to the above question to solve the following inequalities.

(a) $(x-3)(x+2) < 0$ (b) $(x+4)(x-5) \geq 0$ (c) $(2-x)(x+1) \leq 0$

(d) $(1-2x)(1+x) > 0$ (e) $(x-2)(2x-4) \leq 0$ (f) $(-x-2)(x+2) > 0$

Question 14 Solve the following inequalities by sketching an appropriate graph.

(a) $x^2 - 1 \geq 0$ (b) $4 - x^2 > 0$ (c) $x^2 - 5x + 4 \leq 0$

(d) $6 - x - x^2 < 0$ (e) $2x^2 + 7x + 5 < 0$ (f) $1 - x - 6x^2 \leq 0$

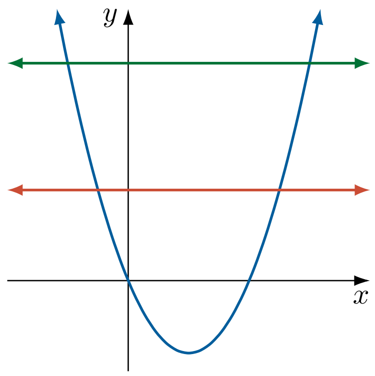
Question 15 [Trick questions]

Solve the following inequalities by sketching an appropriate graph.

(a) $x^2 + 1 > 0$ (b) $x^2 + 4 < 0$ (c) $x^2 \leq 0$

(d) $x^2 + 2x + 2 > 0$ (e) $-x^2 + 2x - 2 \geq 0$ (f) $4x^2 - 4x + 1 \leq 0$

Question 16 The diagram below shows the graph of $y = x^2 - 4x$, $y = 5$ and $y = 12$.



(a) Find the x-intercepts of the parabola, and the intersections with $y = 5$ and $y = 12$.

(b) Hence, use your diagram to solve the following inequalities.

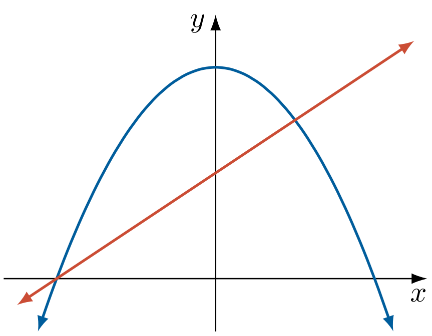
(i) $x^2 - 4x > 0$

(ii) $x^2 - 4x \geq 12$

(iii) $x^2 - 4x \leq 5$

(iv) $5 < x^2 - 4x < 12$

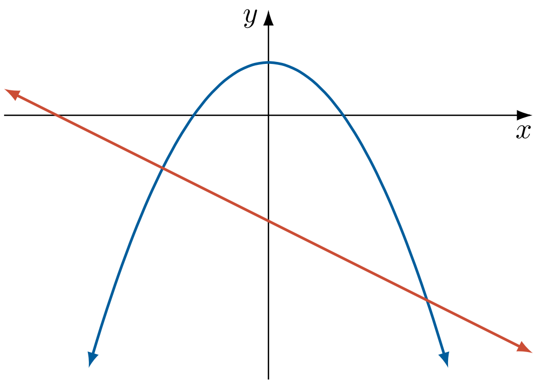
Question 17 The diagram below shows the graph of $y = 4 - x^2$ and $y = x + 2$.



(a) Find where the two graphs intersect.

(b) Hence, solve $4 - x^2 > x + 2$.

Question 18 The diagram below shows a sketch of $y = 2 - x^2$ and $y = -x - 4$.



(a) Find where the two graphs intersect.

(b) Hence, solve $2 - x^2 < -x - 4$.

Question 19 Draw appropriate graphs to solve the following quadratic inequalities.

(a) $(2-x)(x+4) < 0$
(b) $x^2 - 3x + 2 \geq 0$
(c) $x^2 < -2x$
(d) $x^2 \leq x + 6$
(e) $12x - x^2 \geq 0$
(f) $x^2 - 4 \geq -2x + 4$

Question 20

(a) Sketch the graph of $y = x^2 - 2x - 8$

(b) Sketch the graph of $y = |x^2 - 2x - 8|$.

(c) Use your diagram to explain briefly why $|x^2 - 2x - 8| = k$ can never have exactly one solution, for any value of k.

(d) Hence, find the value(s) of k such that $|x^2 - 2x - 8| = k$ has
   (i) no solutions.
   (ii) two solutions.
   (iii) three solutions.
   (iv) four solutions.

Question 21 Use graphing software to sketch the pairs of functions below, and hence state how many solutions there are to the equation $f(x) = g(x)$.

(a) $f(x) = |x-4|$, $g(x) = \frac{1}{2}x$
(b) $f(x) = |x+2| =$, $g(x) = -2x$
(c) $f(x) = x^3 - x$, $g(x) = \cos x$
(d) $f(x) = e^{-x^2}$, $g(x) = x^2$

Question 22 By sketching $y =$ LHS and $y =$ RHS, state the number of solutions to the following equations.

(a) $x = 4 - x^2$
(b) $1 - x^2 = x^2 - 4$
(c) $x = x^3 + 2$
(d) $x = \sqrt{x}$
(e) $x^2 + 1 = \frac{1}{x}$
(f) $1 - x^2 = 3 - 2x$

Question 23 State two appropriate curves that could be used to determine the number of solutions to the following equations.

(a) $x - \cos x = 0$
(b) $x^2 - \ln x = 0$
(c) $e^x - x - 2 = 0$
(d) $x - \sqrt{x} + 1 = 0$
(e) $\ln x - 1 - x = 0$
(f) $1 - \frac{1}{x-1} - x^2 = 0$

Question 24 Determine the number of solutions to the following equations, by drawing appropriate sketches.

(a) $e^x - 1 + x^2 = 0$
(b) $e^x + \cos x - 2 = 0$
(c) $x^2 - \cos x - 1 = 0$

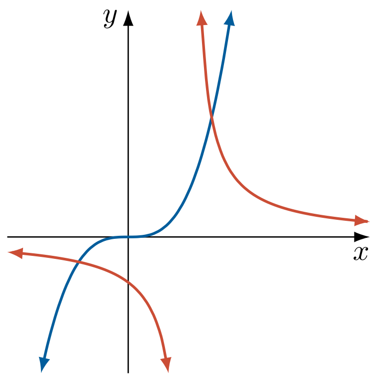
Question 25 Determine the number of solutions to the following equations by drawing appropriate sketches.

(a) $x(x^2 - 1) = 1$
(b) $(x-1)(x^2+1) = 1$
(c) $x^3 - x + 1 = 0$
(d) $x^3 + 2x - 3 = 0$
(e) $x^3 - x^2 - 2 = 0$
(f) $x^4 - x + 1 = 0$

Question 26 Show that the equation $\cos x = e^x$ has infinitely many solutions.

Question 27 [Finding the number of solutions of polynomial equations]

The diagram shows a sketch of $y = x^3$ and $y = \dfrac{1}{x-1}$ on the same set of axes.



(a) How many solutions does $x^3 = \dfrac{1}{x-1}$ have?

(b) How many solutions does $x^3(x-1) = 1$ have?

(c) Hence, how many solutions does $x^4 - x^3 - 1 = 0$ have?

Challenge Problems

Problem 1

(a) Draw a sketch of $y = x^3$ and $y = k - x$ for when $k > 0$ and when $k < 0$.

(b) Hence, show that if $k > 0$ then $x^3 + x - k = 0$ has one positive solution, but if $k < 0$ then $x^3 + x - k = 0$ has one negative solution.

Problem 2 Show that the equation $x^4 - kx - 1 = 0$ will always have two real solutions, and that there will always be one positive and one negative solution.

Problem 3 The equation $x^2 + \ln x = k$ has one real root $x = \alpha$ for all real values of k. Describe the behaviour of α for varying values of k.

Problem 4 Consider the equation $x^3 - kx + 1 = 0$.

(a) Show that if the equation has only one real root, then the root must be negative.

(b) Show that if the equation has a double root, then the double root will be positive.

(c) Show that if the equation has three distinct real roots, then two must be positive and one must be negative.

(d) As $k \to \infty$, one root approaches infinity and another approaches negative infinity. What happens to the third root?

(e) Describe the behaviour of the real root as $k \to -\infty$.

Chapter 1 Review

Transformations

Review

Question 1 In each of the following, $f(x)$ was transformed a certain way for it to become $g(x)$. Describe the transformation.

(a) $f(x) = \sqrt{x},\ g(x) = \sqrt{x+1}$

(b) $f(x) = \sqrt{x},\ g(x) = \sqrt{x} + 2$

(c) $f(x) = \ln(x+1),\ g(x) = \ln(2x+1)$

(d) $f(x) = \dfrac{2}{x-1},\ g(x) = \dfrac{1}{x-1}$

(e) $f(x) = \sin(x),\ g(x) = \sin\left(\dfrac{x}{2}\right)$

(f) $f(x) = e^x + 1,\ g(x) = e^x - 2$

(g) $f(x) = (x+1)^2,\ g(x) = (x-3)^2$

(h) $f(x) = 4x^2 - 1,\ g(x) = x^2 - 1$

Question 2 Write down the equation of the new curve when the following curves have been transformed in the following ways.

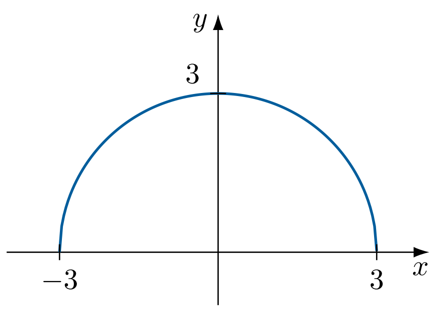
(a) $f(x) = (x-2)^3$, translate left 3 units.

(b) $f(x) = (x-2)^3$, translate right 2 units.

(c) $f(x) = 3x + 2$, translate left 3 units.

(d) $f(x) = 3x + 2$, translate right 4 units.

Question 3 The diagram below shows the graph of $y = f(x)$.



Describe the transformation and sketch the graph of the following.

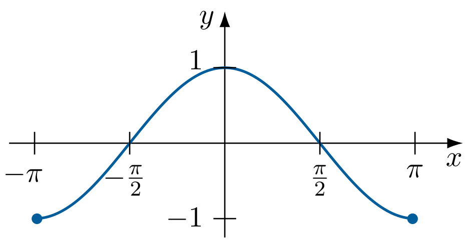
(a) $y = 3f(3x)$

(b) $y = \dfrac{1}{3}f\left(\dfrac{x}{3}\right)$

(c) $y = \dfrac{1}{3}f(3x)$

(d) $y = 3f\left(\dfrac{x}{3}\right)$

Question 4 The diagram below shows the graph of $y = \cos x$ in the domain $x \in [-\pi, \pi]$. By applying the appropriate dilations, sketch the following graphs.



(a) $y = 2\cos(2x)$

(b) $y = \frac{1}{2}\cos(2x)$

(c) $y = 2\cos\left(\frac{x}{2}\right)$

(d) $y = \frac{1}{2}\cos\left(\frac{x}{2}\right)$

Question 5 Consider the graph of $y = \sqrt{x}$. Find the equation of the graph if $y = \sqrt{x}$ is

(a) translated left by 2 units, then stretched horizontally by a factor of 3.

(b) stretched horizontally by a factor of 3, then translated left by 2 units.

(c) translated right by 1 unit, then squashed horizontally by a factor of 2.

(d) squashed horizontally by a factor of 2, then translated right by 1 unit.

Question 6 The curve $y = f(x)$ is translated and dilated, not necessarily in that particular order, to obtain the final curve $y = f(ax + b)$. Find $f(ax + b)$.

(a) translated left by 1 unit, squashed horizontally by a factor of 2.

(b) squashed horizontally by a factor of 2, translated left by 1 unit

(c) translated right by 2 units, stretched horizontally by a factor of 3.

(d) stretched horizontally by a factor of 3, translated right by 2 units,

Question 7 The curve $y = f(x)$ is translated and reflected, not necessarily in that particular order, to obtain the final curve $y = f(ax + b)$. Find $f(ax + b)$.

(a) translated right by 2 units, then reflected across the y-axis.

(b) reflected across the y-axis, then translated right by 2 units.

(c) translated left by 3 units, then reflected across the y-axis.

(d) reflected across the y-axis, then translated left by 3 units.

Question 8 Describe a suitable sequence of transformations that can turn $f(x)$ into $g(x)$ below.

(a) $f(x) = \sin(x),\ g(x) = \sin\left(2x + \frac{\pi}{3}\right)$

(b) $f(x) = \ln(x),\ g(x) = \ln(3x - 6)$

(c) $f(x) = \sqrt{x},\ g(x) = \sqrt{3 - x}$

(d) $f(x) = \tan(x),\ g(x) = \tan\left(\frac{\pi}{4} - x\right)$

Question 9 Describe a suitable sequence of transformations that can turn $f(x)$ into $g(x)$ below.

(a) $f(x) = \cos(x),\ g(x) = \cos\left(\frac{\pi}{3} - 2x\right)$

(b) $f(x) = \ln(x),\ g(x) = \ln\left(2 - \frac{x}{3}\right)$

Question 10 Let $f(x) = \dfrac{x-1}{x+1}$

(a) Find any intercepts with the coordinate axes.

(b) State the equation of the vertical and horizontal asymptote.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 11 Let $f(x) = \dfrac{x^2}{x^2 + 9}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

(d) State where the curve is above and below the x-axis.

(e) Determine whether the function is even, odd, or neither.

(f) Hence, sketch the graph of $y = f(x)$.

Question 12 Let $f(x) = \dfrac{x^2 + 1}{x^2 - 4}$.

(a) Find any intercepts with the coordinate axes.

(b) Find the equation of any horizontal or vertical asymptotes.

(c) Describe the behaviour of the curve as $x \to \pm\infty$.

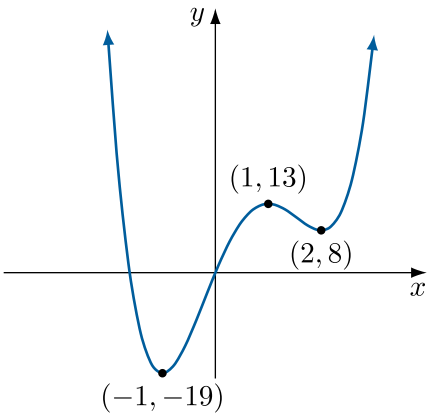
(d) Determine whether the function is even, odd, or neither.

(e) Hence, sketch the graph of $y = f(x)$.

Question 13 Sketch the following, labelling all important features.

(a) $y = \dfrac{\sqrt{x}+1}{\sqrt{x}-1}$ (b) $y = \dfrac{e^x-2}{e^x+2}$

Question 14 The diagram below shows the graph of a function $y = f(x)$. Find the value(s) of k such that $f(x) = k$ has



(a) one solution. (b) two solutions.

(c) three solutions. (d) four solutions.

Question 15 Solve the following quadratic inequalities.

(a) $(x+3)(x-2) \geq 0$ (b) $(2-3x)(x+4) > 0$ (c) $x^2+9x+20 \leq 0$

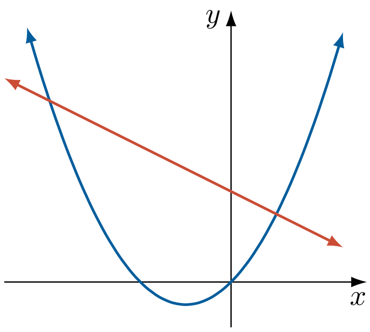
(d) $6+x-x^2 \geq 0$ (e) $4x^2+12x-7 > 0$ (f) $2+3x-9x^2 > 0$

Question 16 Solve the following quadratic inequalities.

(a) $x^2+4x+5 > 0$ (b) $x^2+6x+10 < 0$ (c) $4x^2+4x+1 \leq 0$

Question 17 By drawing an appropriate graph, solve the inequality $-5 \leq x^2-6x \leq 7$.

Question 18 The diagram below shows a sketch of $y = x^2+2x$ and $y = 4-x$.



(a) Find where the two graphs intersect. (b) Hence, solve $x^2+2x > 4-x$.

Question 19 Determine the number of solutions to the following equations by drawing appropriate sketches.

(a) $x^2 - \sin x = 0$ (b) $2x - \sin x - 1 = 0$ (c) $e^x + x - 2 = 0$

(d) $e^{-x} + x^2 - 2 = 0$ (e) $x^2 - \ln x - 4 = 0$ (f) $\ln x + x + 4 = 0$

Question 20 Determine the number of solutions to the following equations by drawing appropriate sketches.

(a) $x^3 - x + 2 = 0$ (b) $x^3 - x^2 - 1 = 0$ (c) $x^4 - x^3 + x - 2 = 0$