

5A

$$\begin{aligned} \text{Q1 a) } A &= \int_0^2 y^2 dy = \left[\frac{y^3}{3} \right]_0^2 \\ &= \frac{8}{3} \\ \text{b) } A &= \int_1^4 y^{\frac{3}{2}} dy = \left[2y^{\frac{5}{2}} \right]_1^4 \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{c) } A &= \int_0^1 2y dy = \left[y^2 \right]_0^1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{d) } A &= \int_0^8 y^{\frac{1}{3}} dy = \frac{3}{4} \left[y^{\frac{4}{3}} \right]_0^8 \\ &= \frac{3}{4} \times 16 = 12 \end{aligned}$$

$$\begin{aligned} \text{e) } A &= \int_1^9 (y-1)^{\frac{2}{3}} dy = \frac{3}{4} \left[(y-1)^{\frac{4}{3}} \right]_1^9 \\ &= \frac{3}{4} \times 16 = 12 \end{aligned}$$

$$\begin{aligned} \text{f) } A &= \int_1^2 4 - y^2 dy = \left[4y - \frac{y^3}{3} \right]_1^2 \\ &= 8 - \frac{8}{3} - 4 + \frac{1}{3} \\ &= \frac{5}{3} \end{aligned}$$

$$\begin{aligned} \text{g) } A &= \int_1^2 e^y dy = \left[e^y \right]_1^2 \\ &= e^2 - e \end{aligned}$$

$$\begin{aligned} \text{h) } A &= \int_1^9 y^{-2} dy = \left[-\frac{1}{y} \right]_1^9 \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{Q3 a) } A &= \int y dx = \int_0^4 \frac{2}{3} (4-x)^{\frac{3}{2}} dx \\ &= \frac{2}{3} \left[-\frac{2}{5} (4-x)^{\frac{5}{2}} \right]_0^4 = 2 \times 8 \\ &= 16 \end{aligned}$$

$$\begin{aligned} \text{b) } A &= \int x dy = \int_0^6 4 - \frac{y^2}{4} dy \\ &= \left[4y - \frac{y^3}{12} \right]_0^6 = 24 - 8 \\ &= 16 \end{aligned}$$

$$\begin{aligned} \text{Q4 a) } A_1 &= \int_0^{16} x^{\frac{1}{2}} dx = \frac{2}{3} \left[x^{\frac{3}{2}} \right]_0^{16} \\ &= \frac{2}{3} \times 64 \\ &= \frac{128}{3} \end{aligned}$$

$$\begin{aligned} \text{b) } A_2 &= \int_0^4 y^2 dy = \left[\frac{y^3}{3} \right]_0^4 \\ &= \frac{64}{3} \end{aligned}$$

$$\begin{aligned} \text{c) } A_1 + A_2 &= 64 \text{ it is the area of the rectangle enclosed by} \\ &0 \leq x \leq 16 \text{ and } 0 \leq y \leq 4 \end{aligned}$$

$$\begin{aligned} \text{Q5 shaded region} &= \text{area of rectangle} - \text{area bounded by curve and axis} \\ &= 8 - \int_0^4 (4-x)^{\frac{3}{2}} dx \quad \text{OR} \quad 8 - \int_0^2 4 - y^2 dy \\ &= 8 - \left[-\frac{2}{5} (4-x)^{\frac{5}{2}} \right]_0^4 = 8 - \left(\frac{2}{5} \times 8 \right) \quad = 8 - \left[4y - \frac{y^3}{3} \right]_0^2 \\ &= \frac{8}{5} \quad = 8 - \left(8 - \frac{8}{3} \right) \\ & \quad = \frac{8}{3} \end{aligned}$$

Q6

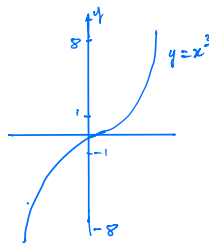
Q7

Q8

$$\begin{aligned} \text{Q9 a) } A &= \int_1^8 y^{\frac{1}{3}} dy = \frac{3}{4} \left[y^{\frac{4}{3}} \right]_1^8 \\ &= \frac{3}{4} (16 - 1) \\ &= \frac{45}{4} \end{aligned}$$

$$\begin{aligned} \text{b) } A &= -\int_{-8}^{-1} y^{\frac{1}{3}} dy = -\frac{3}{4} \left[y^{\frac{4}{3}} \right]_{-8}^{-1} \\ &= -\frac{3}{4} (1 - 16) \\ &= \frac{45}{4} \end{aligned}$$

$$\begin{aligned} \text{c) } A &= 2 \int_0^8 y^{\frac{1}{3}} dy = 2 \times \left[\frac{3}{4} y^{\frac{4}{3}} \right]_0^8 \\ &= 2 \times \frac{3}{4} \times 16 \\ &= 24 \end{aligned}$$



Q10 a) $A = \int_0^4 (4y - y^2) dy$
 $= \left[2y^2 - \frac{y^3}{3} \right]_0^4 = 32 - \frac{64}{3}$
 $= \frac{32}{3}$

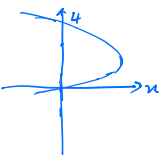
b) $A = - \int_0^1 y^2 - y dy$
 $= - \left[\frac{y^3}{3} - \frac{y^2}{2} \right]_0^1 = - \left(\frac{1}{3} - \frac{1}{2} \right)$
 $= \frac{1}{6}$

c) $A = \int_2^6 -y^2 + 8y - 12 dy$
 $= \left[-\frac{y^3}{3} + 4y^2 - 12y \right]_2^6$
 $= -\frac{6^3}{3} + 4 \times 36 - 72 - \left(-\frac{2^3}{3} + 16 - 24 \right)$
 $= \frac{32}{3}$

d) $A = \int_2^5 -y^2 + 7y - 10 dy$
 $= \left[-\frac{y^3}{3} + \frac{7y^2}{2} - 10y \right]_2^5$
 $= -\frac{125}{3} + \frac{7}{2} \times 25 - 50 - \left(-\frac{8}{3} + 14 - 20 \right)$
 $= \frac{9}{2}$

$-y^2 + 8y - 12 = 0$
 $y^2 - 8y + 12 = 0$
 $(y - 6)(y - 2) = 0$
 $y = 6 \quad y = 2$

$-y^2 + 7y - 10 = 0$
 $(y - 5)(y - 2) = 0$
 $y = 5 \quad y = 2$



Q11 We need to make x the subject in these questions and use $A = \int x dy$.

a) $A = \int_0^{\ln 3} 3 - e^y dy = \left[3y - e^y \right]_0^{\ln 3} \quad x = 3 - e^y$
 $= 3 \ln 3 - 3 + e^0$
 $= 3 \ln 3 - 2$

b) $A = - \int_0^{\ln 4} e^{\frac{y}{2}} - 2 dy = - \left[2e^{\frac{y}{2}} - 2y \right]_0^{\ln 4} \quad x = e^{\frac{y}{2}} - 2$
 $= - (2 \times 2 - 2 \ln 4 - 2)$
 $= 4 \ln 2 - 2$

c) $A = \int_0^{\frac{\pi}{2}} \cos y dy = \left[\sin y \right]_0^{\frac{\pi}{2}} \quad x = \cos y$
 $= 1$

d) $A = - \int_0^{\frac{\pi}{4}} \tan y - 1 dy \quad x = \tan y - 1$
 $= - \left[-\ln |\cos y| - y \right]_0^{\frac{\pi}{4}}$
 $= - \left(-\ln \frac{1}{\sqrt{2}} - \frac{\pi}{4} \right)$
 $= \frac{\pi}{4} - \frac{1}{2} \ln 2$

Q12

Q13 a) Area of region $= ex1 - \int_0^1 e^y dy$
 $= e - \left[e^y \right]_0^1 = e - (e - 1)$
 $= 1$

b) Area of region $= \frac{\pi}{2} \times 2 - \int_0^{\frac{\pi}{2}} 2 \sin x dx$
 $= \pi - \left[-2 \cos x \right]_0^{\frac{\pi}{2}}$
 $= \pi + 2(0 - 1)$
 $= \pi - 2$

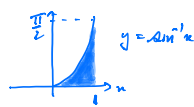
c) Area of region $= 2x \ln 3 - \int_0^{\ln 3} e^y - 1 dy$
 $= 2 \ln 3 - \left[e^y - y \right]_0^{\ln 3}$
 $= 2 \ln 3 - (3 - \ln 3 - 1)$
 $= 3 \ln 3 - 2$

d) Area of region $= \frac{\pi}{4} \times 1 - \int_0^{\frac{\pi}{4}} \tan y dy$
 $= \frac{\pi}{4} + \left[\ln \cos y \right]_0^{\frac{\pi}{4}} = \frac{\pi}{4} + \ln \frac{1}{\sqrt{2}}$
 $= \frac{\pi}{4} - \frac{1}{2} \ln 2$

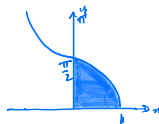
e) Area of region $= 2x \frac{3\pi}{2} - \int_0^{\frac{3\pi}{2}} 2 \sin \frac{y}{2} dy \quad x = 2 \sin \frac{y}{2}$
 $= 3\pi + 6 \left[\cos \frac{y}{2} \right]_0^{\frac{3\pi}{2}} = 3\pi + 6(0 - 1)$
 $= 3\pi - 6$

f) Area of region $= 1 \times \pi - \left| \int_{\frac{\pi}{2}}^{\pi} \cos y dy \right| = \pi - \left| \left[\sin y \right]_{\frac{\pi}{2}}^{\pi} \right|$
 $= \pi - 1$

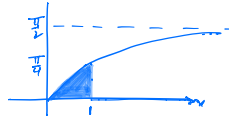
$$\begin{aligned} \text{Q14 a) } A &= \frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \sin y \, dy \\ &= \frac{\pi}{2} + [\cos y]_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} - 1 \end{aligned}$$



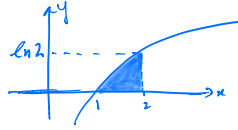
$$\begin{aligned} \text{b) } A &= \int_0^{\frac{\pi}{2}} \cos y \, dy \\ &= [\sin y]_0^{\frac{\pi}{2}} \\ &= 1 \end{aligned}$$



$$\begin{aligned} \text{c) } A &= \frac{\pi}{4} - \int_0^{\frac{\pi}{4}} \tan y \, dy \\ &= \frac{\pi}{4} + [\ln \cos y]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4} + \ln \frac{1}{\sqrt{2}} \\ &= \frac{\pi}{4} - \frac{1}{2} \ln 2 \end{aligned}$$



$$\begin{aligned} \text{d) } A &= 2 \ln 2 - \int_0^{\ln 2} e^y \, dy \\ &= 2 \ln 2 - [e^y]_0^{\ln 2} \\ &= 2 \ln 2 - (2 - 1) \\ &= 2 \ln 2 - 1 \end{aligned}$$



$$\begin{aligned} \text{Q15 a) } y &= 8y - y^2 \\ y(y - 7) &= 0 \\ y &= 0, \quad y = 7 \\ (0, 0) \quad (7, 7) \end{aligned}$$

$$\begin{aligned} \text{b) } A &= \frac{1}{2} \times 7 \times 7 + \int_7^8 (8y - y^2) \, dy \\ &= \frac{49}{2} + \left[4y^2 - \frac{y^3}{3} \right]_7^8 = \frac{49}{2} + (256 - \frac{8^3}{3} - 4 \times 7^2 + \frac{7^3}{3}) \\ &= \frac{49}{2} + \frac{11}{3} = \frac{169}{6} \end{aligned}$$

$$\begin{aligned} \text{c) Region B} &= \int_0^8 (8y - y^2) \, dy - \text{region A} \\ &= \left[4y^2 - \frac{y^3}{3} \right]_0^8 - \frac{169}{6} \\ &= \frac{343}{6} \end{aligned}$$

$$\begin{aligned} \text{OR} &= \int_0^7 (8y - y^2 - y) \, dy = \int_0^7 (7y - y^2) \, dy \\ &= \left[\frac{7y^2}{2} - \frac{y^3}{3} \right]_0^7 = \frac{343}{6} \end{aligned}$$

$$\begin{aligned} \text{Q16 } y+5 &= -y^2 - 6y - 5 && \text{when } x=0 \\ y^2 + 7y + 10 &= 0 && 0 = -y^2 - 6y - 5 \\ (y+2)(y+5) &= 0 && (y+1)(y+5) = 0 \\ y &= -2 \text{ or } y = -5 && y = -1 \text{ or } y = -5 \\ (3, -2) &&& (0, -5) \end{aligned}$$

$$\begin{aligned} \text{Shaded region} &= \int_{-5}^{-1} (-y^2 - 6y - 5) - (y+5) \, dy \\ &= \left[-\frac{y^3}{3} - \frac{7y^2}{2} - 10y \right]_{-5}^{-1} \\ &= \left[\frac{4}{3} + \frac{7}{2} + 10 \right]_{-2}^{-5} \\ &= -\frac{125}{3} + \frac{7}{2} \times 25 - 50 - \left(-\frac{8}{3} + 14 - 20 \right) \\ &= \frac{9}{2} \end{aligned}$$

$$\begin{aligned} \text{P1 Area enclosed by y-axis and } y &= \sin \frac{\pi x}{2} \\ &= 1 \times 1 - \int_0^1 \sin \frac{\pi x}{2} \, dx = 1 + \frac{2}{\pi} \left[\cos \frac{\pi x}{2} \right]_0^1 \\ &= 1 + \frac{2}{\pi} (-1) \end{aligned}$$

$$\begin{aligned} \text{Shaded region} &= 1 \times 1 - 2 \left(1 - \frac{2}{\pi} \right) = 1 - 2 + \frac{4}{\pi} \\ &= \frac{4}{\pi} - 1 \end{aligned}$$

$$\begin{aligned} \text{P2 a) Area B} &= \int_0^1 x^n \, dx = \left[\frac{x^{n+1}}{n+1} \right]_0^1 \\ &= \frac{1}{n+1} \end{aligned}$$

$$\text{Area A} = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

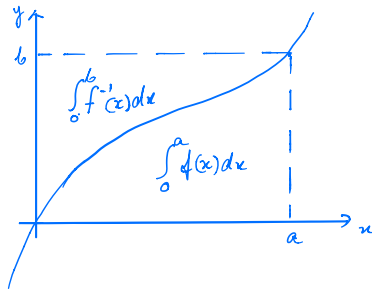
$$\begin{aligned} \text{If Area A : Area B} &= \frac{n}{n+1} : \frac{1}{n+1} = 2 : 1 \\ n &= 2 \end{aligned}$$

$$\begin{aligned} \text{b) If } \frac{n}{n+1} : \frac{1}{n+1} &= 3 : 1 \\ n &= 3 \end{aligned}$$

$$\begin{aligned} \text{c) If } n : 1 &= p : q \\ n &= \frac{p}{q} \end{aligned}$$

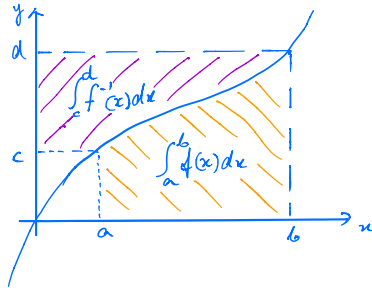
P3

- The two areas add to become the full rectangle, which has area ab .



P4

- The sum of the two areas as shown in diagram become $bd - ac$ ie. subtract small rectangle from large rectangle.



Q1 a) cylinder $V = \pi r^2 h = \pi \times 3^2 \times 4$

$$= 36\pi$$

b) cylinder $V = \pi \times 4^2 \times 3$

$$= 48\pi$$

c) cone $V = \frac{1}{3} \times \pi r^2 h = \frac{1}{3} \times \pi \times 9^2 \times 3$

$$= 81\pi$$

d) cylinder with cone removed from inside it

$$V = \pi \times 3^2 \times 9 - \frac{1}{3} \pi \times 3^2 \times 9$$

$$= 54\pi$$

e) cylinder with cylinder removed from inside it

$$V = \pi \times 4^2 \times 2 - \pi \times 1^2 \times 2$$

$$= 30\pi$$

Q2 Using formula $V = \pi \int y^2 dx$

a) $V = \pi \int_0^2 x^2 dx = \pi \int_0^2 \frac{x^3}{3} dx$

$$= \frac{8}{3}\pi$$

b) $V = \pi \int_0^1 x^4 dx = \pi \left[\frac{x^5}{5} \right]_0^1$

$$= \frac{\pi}{5}$$

c) $V = 2\pi \int_0^3 81 - 18x^2 + x^4 dx$
 $= 2\pi \left[81x - 6x^3 + \frac{x^5}{5} \right]_0^3$
 $= \frac{1296\pi}{5}$

d) $V = \pi \int_0^1 x^2 - 2x^3 + x^4 dx$
 $= \pi \left[\frac{x^3}{3} - \frac{1}{2}x^4 + \frac{x^5}{5} \right]_0^1$
 $= \frac{\pi}{30}$

e) $V = \pi \int_0^4 x dx = \pi \left[\frac{x^2}{2} \right]_0^4$
 $= 8\pi$

f) $V = \pi \int_1^2 \frac{1}{x^2} dx = \pi \left[-\frac{1}{x} \right]_1^2$
 $= \frac{\pi}{2}$

g) $V = \pi \int_0^3 \frac{1}{x+1} dx = \pi [\ln|x+1|]_0^3$
 $= \pi \ln 4 = 2\pi \ln 2$

h) $V = \pi \int_1^3 \frac{1}{(2x+1)^2} dx = \pi \left[\frac{1}{2(2x+1)} \right]_1^3$
 $= \pi \left[-\frac{1}{14} + \frac{1}{6} \right]$
 $= \frac{2\pi}{21}$

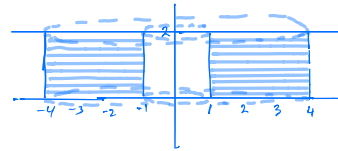
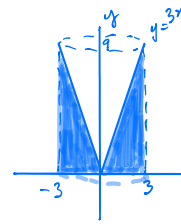
i) $V = \pi \int_0^2 x^4(2-x) dx = \pi \left[\frac{2x^5}{5} - \frac{x^6}{6} \right]_0^2$
 $= \pi \left(\frac{64}{5} - \frac{64}{6} \right)$
 $= \frac{32\pi}{15}$

j) $V = \pi \int_0^2 16x^2 - 8x^4 + x^6 dx$
 $= \pi \left[\frac{16}{3}x^3 - \frac{8}{5}x^5 + \frac{x^7}{7} \right]_0^2$
 $= \frac{1024\pi}{105}$

Q3 a) $V = \pi \int_0^{\frac{1}{2}} \frac{1}{4+9x^2} dx$
 $= \frac{\pi}{6} \left[\tan^{-1} \frac{3x}{2} \right]_0^{\frac{1}{2}} = \frac{\pi}{6} \times \frac{\pi}{4}$
 $= \frac{\pi^2}{24}$

b) $V = \pi \int_{\frac{1}{3}}^{\sqrt{3}} \frac{1}{1+x^2} dx = \pi \left[\tan^{-1} x \right]_{\frac{1}{3}}^{\sqrt{3}}$
 $= \pi \left(\frac{\pi}{3} - \frac{\pi}{6} \right)$
 $= \frac{\pi^2}{6}$

c) $V = 2\pi \int_0^1 \frac{1}{\sqrt{4-x^2}} dx = 2\pi \left[\sin^{-1} \frac{x}{2} \right]_0^1$
 $= 2\pi \times \frac{\pi}{6} = \frac{\pi^2}{3}$



$$d) V = \pi \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-4x^2}} dx = \frac{\pi}{2} [\sin^{-1} 2x]_0^{\frac{\pi}{2}} \\ = \frac{\pi}{2} \times \frac{\pi}{6} = \frac{\pi^2}{12}$$

$$Q4) a) V = \pi \int_0^{\pi} \sin^2 x dx = \frac{\pi}{2} \int_0^{\pi} 1 - \cos 2x dx \quad \begin{aligned} \cos 2x &= 1 - 2\sin^2 x \\ &= 2\cos^2 x - 1 \end{aligned} \\ = \frac{\pi}{2} \left[x - \frac{1}{2} \sin 2x \right]_0^{\pi} \\ = \frac{\pi}{2} (\pi) = \frac{\pi^2}{2}$$

$$b) V = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 x dx = \frac{\pi}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 1 + \cos 2x dx \\ = \frac{\pi}{2} \left[x + \frac{1}{2} \sin 2x \right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ = \frac{\pi}{2} \left(\frac{\pi}{3} + \frac{1}{2} \times \frac{\sqrt{3}}{2} - \frac{\pi}{6} - \frac{1}{2} \times \frac{\sqrt{3}}{2} \right) \\ = \frac{\pi^2}{12}$$

$$c) V = 9\pi \int_0^{\frac{\pi}{2}} \sin^2 2x dx = \frac{9\pi}{2} \int_0^{\frac{\pi}{2}} 1 - \cos 4x dx \\ = \frac{9\pi}{2} \left[x - \frac{1}{4} \sin 4x \right]_0^{\frac{\pi}{2}} \\ = \frac{9\pi}{2} \times \frac{\pi}{2} \\ = \frac{9\pi^2}{4}$$

$$d) V = 16\pi \int_0^{2\pi} \sin^2 \frac{x}{2} dx = 8\pi \int_0^{2\pi} 1 - \cos x dx \\ = 8\pi \left[x - \sin x \right]_0^{2\pi} \\ = 16\pi^2$$

$$e) V = \pi \int_0^{\frac{\pi}{4}} \tan^2 x dx = \pi \int_0^{\frac{\pi}{4}} \sec^2 x - 1 dx \\ = \pi \left[\tan x - x \right]_0^{\frac{\pi}{4}} \\ = \pi \left(1 - \frac{\pi}{4} \right) \\ = \pi - \frac{\pi^2}{4}$$

$$f) V = 4\pi \int_0^{\frac{\pi}{3}} \tan^2 3x dx = 4\pi \int_0^{\frac{\pi}{3}} \sec^2 3x - 1 dx \\ = 4\pi \left[\frac{1}{3} \tan 3x - x \right]_0^{\frac{\pi}{3}} \\ = 4\pi \left[\frac{\sqrt{3}}{3} - \frac{\pi}{9} \right] \\ = 4\pi \left(\frac{\sqrt{3}}{3} - \frac{\pi}{9} \right)$$

$$Q5) a) V = \pi \int_0^2 e^{-x} dx = -\pi [e^{-x}]_0^2 \\ = \pi \left(1 - \frac{1}{e^2} \right)$$

$$b) V = \pi \int_{-\ln 4}^{\ln 4} e^{-x} dx = -\pi [e^{-x}]_{-\ln 4}^{\ln 4} \\ = \pi \left(4 - \frac{1}{4} \right) \\ = \frac{15\pi}{4}$$

$$Q6) a) \int_0^{\frac{\pi}{2}} \tan 3x dx = -\frac{1}{3} [\ln \cos 3x]_0^{\frac{\pi}{2}} \\ = -\frac{1}{3} \ln \frac{1}{2} \\ = \frac{1}{3} \ln 2$$

represents the area of the shaded region

$$b) \pi \int_0^{\frac{\pi}{2}} \tan^2 3x dx = \pi \int_0^{\frac{\pi}{2}} \sec^2 3x - 1 dx \\ = \pi \left[\frac{1}{3} \tan 3x - x \right]_0^{\frac{\pi}{2}} \\ = \pi \left[\frac{1}{3} - \frac{\pi}{12} \right] = \frac{\pi}{12} (4 - \pi)$$

represents the volume if the shaded region is rotated about the x axis.

$$Q7) V = \pi \int_0^4 (4-x) dx = 9\pi \left[4x - \frac{x^2}{2} \right]_0^4 \\ = 9\pi (16 - 8) \\ = 72\pi$$

$$Q8) V = 2\pi \int_0^2 e^{2x} + 2 + e^{-2x} dx \\ = \frac{4\pi}{2} \left[\frac{1}{2} e^{2x} + 2x - \frac{1}{2} e^{-2x} \right]_0^2 \\ = \frac{\pi}{2} \left(\frac{1}{2} e^4 + 4 - \frac{1}{2} e^{-4} - \frac{1}{2} + \frac{1}{2} \right) \\ = \frac{\pi}{4} (e^4 + 8 - e^{-4})$$

$$Q9) V = \pi \int_0^h r^2 dx = \pi [r^2 x]_0^h \\ = \pi r^2 h \quad \text{Volume formed is a cylinder as } y=r \text{ is rotated about the } x \text{ axis in domain } [0, h]$$

Q10 a) x_0 is height of cone i.e. h
 y_0 is radius of cone i.e. r So
 $(x_0, y_0) = (h, r)$

b) $r = kh$

$$k = \frac{r}{h}$$

c) $V = \pi \int_0^h k^2 x^2 dx = \pi \left[k^2 \frac{x^3}{3} \right]_0^h$
 $= \frac{\pi}{3} k^2 h^3 = \frac{\pi}{3} \frac{r^2}{h^2} h^3$
 $= \frac{\pi}{3} r^2 h$

Q11 $V = 2\pi \int_0^r (r^2 - x^2) dx$
 $= 2\pi \left[r^2 x - \frac{x^3}{3} \right]_0^r$
 $= \frac{4}{3} \pi r^3$

Q12 a) $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$
 $= \frac{a^2 - x^2}{a^2}$
 $y^2 = \frac{b^2}{a^2} (a^2 - x^2)$

b) $V = 2\pi \int_0^a \frac{b^2}{a^2} (a^2 - x^2) dx$
 $= \frac{2b^2\pi}{a^2} \left[a^2 x - \frac{x^3}{3} \right]_0^a$
 $= \frac{4}{3} \frac{b^2}{a^2} a^3 \pi$
 $= \frac{4}{3} \pi a b^2$

c) The sphere is a special case of the ellipsoid when
 $a = b = r$ so
 $V = \frac{4}{3} \pi r^3$

Q13 Solid formed when blue region is rotated about the x axis
 $V = \pi \int_{\frac{1}{a}}^1 \frac{4}{x} dx = 4\pi [\ln x]_{\frac{1}{a}}^1$
 $= 4\pi \ln a$

Solid formed when pink region is rotated about the x axis
 $V = \pi \int_1^a \frac{4}{x} dx = 4\pi [\ln x]_1^a$
 $= 4\pi \ln a$

Q14

Q15

Q16 $V = \pi \int_0^1 (2^2 - 4x^4) dx$
 $= \pi \left[4x - \frac{4}{5} x^5 \right]_0^1 = \pi \left(4 - \frac{4}{5} \right)$
 $= \frac{16\pi}{5}$

Q17 a) $V = \pi \int_1^2 \left(1 - \frac{4}{x} + \frac{4}{x^2} \right) dx$
 $= \pi \left[x - 4 \ln x - \frac{4}{x} \right]_1^2$
 $= \pi [2 - 4 \ln 2 - 2 - (1 - 4)]$
 $= \pi (3 - 4 \ln 2)$
 $V = \pi \int_{-1}^0 \frac{4}{(1-y)^2} - 1 dy$
 $= \pi \left[\frac{4}{1-y} - y \right]_{-1}^0 = \pi (4 - 2 - 1)$
 $= \pi$

$$\frac{2}{x} = 1 - y$$

$$x = \frac{2}{1-y}$$

P1 $V_1 = \pi \int_0^1 x^4 dx = V_2 = \pi \int_1^k x^4 dx$
 $\left[\frac{x^5}{5} \right]_0^1 = \left[\frac{x^5}{5} \right]_1^k$
 $\frac{1}{5} = \frac{k^5}{5} - \frac{1}{5}$
 $2 = k^5$
 $k = \sqrt[5]{2}$

P2 a) $V_1 = \pi \int_1^a \frac{1}{x^2} dx = V_2 = \pi \int_a^b \frac{1}{x^2} dx$
 $\left[-\frac{1}{x} \right]_1^a = \left[-\frac{1}{x} \right]_a^b$

$$1 - \frac{1}{a} = \frac{1}{a} - \frac{1}{b}$$

$$\frac{2}{a} = 1 + \frac{1}{b}$$

$$\frac{1}{b} = \frac{2-a}{a}$$

$$b = \frac{a}{2-a}$$

b) see answer

$$Q1 \ a) \ V = \pi \int_0^5 (2y - 1) dy = \pi [y^2 - y]_0^5$$

$$= 20\pi$$

$$b) \ V = \pi \int_0^2 y^4 + 2y^2 + 1 dy = \pi \left[\frac{y^5}{5} + \frac{2}{3}y^3 + y \right]_0^2$$

$$= \pi \left(\frac{32}{5} + \frac{16}{3} + 2 \right)$$

$$= \frac{206\pi}{15}$$

$$c) \ V = \pi \int_0^4 4y^2 dy = \pi \left[\frac{4}{3}y^3 \right]_0^4$$

$$= \frac{4}{3}\pi$$

$$d) \ V = \pi \int_0^4 y dy = \pi \left[\frac{y^2}{2} \right]_0^4$$

$$= 8\pi$$

$$e) \ V = \pi \int_0^1 (1 - y) dy = \pi \left[y - \frac{y^2}{2} \right]_0^1$$

$$= \frac{\pi}{2}$$

$$f) \ V = \pi \int_1^2 y^4 dy = \pi \left[\frac{y^5}{5} \right]_1^2$$

$$= \frac{31\pi}{5}$$

$$g) \ V = \pi \int_1^2 e^{2y} dy = \frac{\pi}{2} [e^{2y}]_1^2$$

$$= \frac{\pi}{2} (e^4 - e^2)$$

$$h) \ V = \pi \int_1^4 \frac{1}{y^3} dy = -\frac{\pi}{2} \left[\frac{1}{y^2} \right]_1^4$$

$$= \frac{\pi}{2} \left(1 - \frac{1}{16} \right)$$

$$= \frac{21\pi}{64}$$

$$Q2 \ V = \pi \int_0^9 (y + 6\sqrt{y} + 9) dy = \pi \left[\frac{y^2}{2} + 4y^{3/2} + 9y \right]_0^9$$

$$= \frac{489\pi}{2}$$

$$\sqrt{y} = x - 3$$

$$(\sqrt{y} + 3)^2 = x^2$$

$$Q3 \ V = \pi \int_0^6 \left(16 - \frac{8y^2}{9} + \frac{y^4}{81} \right) dy = \pi \left[16y - \frac{8}{27}y^3 + \frac{y^5}{405} \right]_0^6$$

$$= \frac{256\pi}{5}$$

$$\left(\frac{4}{3} \right)^2 = 4 - x$$

$$x = 4 - \frac{y^2}{9}$$

$$Q4 \ V = \pi \int_0^2 (4y^2 - 4y^3 + y^4) dy = \pi \left[\frac{4}{3}y^3 - y^4 + \frac{y^5}{5} \right]_0^2$$

$$= \frac{16\pi}{15}$$

$$Q5 \ a) \ V = \pi \int_0^2 4y^2 dy = \pi \left[\frac{4}{3}y^3 \right]_0^2$$

$$= \frac{32\pi}{3}$$

$$b) \ V = \frac{1}{2}\pi r^2 h \quad h=2 \quad r=4$$

$$= \frac{1}{2}\pi \times 4^2 \times 2$$

$$= \frac{32\pi}{3}$$

$$Q6 \ a) \ \text{Eq of line is } y - 5 = -2x$$

$$2x + y = 5$$

$$\text{Eq of circle } x^2 + y^2 = 25$$

$(4, -3)$ satisfy both equations hence intersect at this point

$$b) \ V = \pi \int_{-3}^5 \left(25 - 10y + y^2 \right) dy$$

$$= \frac{\pi}{3} \left[25y - 5y^2 + \frac{y^3}{3} \right]_{-3}^5$$

$$= \frac{\pi}{3} \left(\frac{125}{3} - (-75 - 45 - 9) \right)$$

$$= \frac{128\pi}{3}$$

$$c) \ V_{\text{con as perc. of } V_{\text{sphere}}} = \frac{128\pi}{3} \times \frac{3}{4\pi \times 125} \times \frac{100}{1}$$

$$= 25.6\%$$

$$Q7 \ a) \ V = \frac{2}{3}\pi r^3$$

$$= \frac{2}{3}\pi \times 1000$$

$$= \frac{2000\pi}{3}$$

$$y^2 = 100 - x^2$$

$$x^2 = 100 - y^2$$

$$b) \ V_{\text{water}} = \pi \int_{-10}^{-5} (100 - y^2) dy = \pi \left[100y - \frac{y^3}{3} \right]_{-10}^{-5}$$

$$= \pi \left(-500 + \frac{125}{3} + 1000 - \frac{1000}{3} \right)$$

$$= \frac{625\pi}{3}$$

$$\% \text{ capacity} = 31.25\%$$

$$Q8 \ V = 2\pi \int_0^8 (16 + y^2) dy = 2\pi \left[16y + \frac{y^3}{3} \right]_0^8$$

$$= \frac{1792\pi}{3}$$

$$Q9 \quad V = 2\pi \int_0^2 16 - 4y^2 dy = 2\pi \left[16y - \frac{4}{3}y^3 \right]_0^2$$

$$= \frac{128\pi}{3}$$

$$Q10 \quad a) \quad V = \pi \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \sin^2 y dy = \frac{\pi}{2} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} (1 - \cos 2y) dy$$

$$= \frac{\pi}{2} \left[y - \frac{1}{2} \sin 2y \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}}$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} \right)$$

$$= \frac{\pi^2}{4}$$

$$b) \quad V = \pi \times 1^2 \times \frac{\pi}{2} - \frac{\pi^2}{4}$$

$$= \frac{\pi^2}{4}$$

$$c) \quad V = \pi \int_{\frac{\pi}{2}}^{\pi} \cos^2 y dy = \frac{\pi}{2} \int_{\frac{\pi}{2}}^{\pi} (1 + \cos 2y) dy$$

$$= \frac{\pi}{2} \left[y + \frac{1}{2} \sin 2y \right]_{\frac{\pi}{2}}^{\pi}$$

$$= \frac{\pi}{2} \left(\pi - \frac{\pi}{2} \right)$$

$$= \frac{\pi^2}{4}$$

$$d) \quad V = \pi \times 1^2 \times \pi - \frac{\pi^2}{4}$$

$$= \frac{3}{4}\pi^2$$

$$e) \quad V = \pi \int_0^{\frac{\pi}{4}} \tan^2 y dy = \pi \int_0^{\frac{\pi}{4}} (\sec^2 y - 1) dy$$

$$= \pi \left[\tan y - y \right]_0^{\frac{\pi}{4}}$$

$$= \pi \left(1 - \frac{\pi}{4} \right)$$

$$f) \quad V = \pi \times 1^2 \times \frac{\pi}{4} - \pi + \frac{\pi^2}{4}$$

$$= \frac{\pi^2}{2} - \pi$$

$$Q11 \quad a) \quad V = \pi \times 1^2 \times \pi - \pi \int_0^{\pi} \sin^2 \frac{y}{2} dy$$

$$= \pi^2 - \frac{\pi}{2} \int_0^{\pi} (1 - \cos y) dy$$

$$= \pi^2 - \frac{\pi}{2} \left[y - \sin y \right]_0^{\pi}$$

$$= \pi^2 - \frac{\pi}{2} (\pi)$$

$$= \frac{\pi^2}{2}$$

$$b) \quad V = \pi \times \left(\frac{1}{3} \right)^2 \times 4\pi - \pi \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \cos^2 \frac{y}{4} dy$$

$$= \frac{4\pi^2}{9} - \frac{\pi}{18} \int_{2\pi}^{4\pi} (1 + \cos \frac{y}{2}) dy$$

$$= \frac{4\pi^2}{9} - \frac{\pi}{18} \left[y + 2 \sin \frac{y}{2} \right]_{2\pi}^{4\pi}$$

$$= \frac{4\pi^2}{9} - \frac{\pi}{18} (4\pi - 2\pi)$$

$$= \frac{4\pi^2}{9} - \frac{\pi^2}{9} = \frac{3\pi^2}{9} = \frac{\pi^2}{3}$$

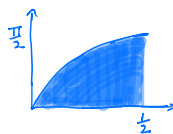
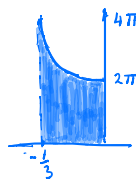
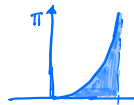
$$c) \quad V = \pi \times \left(\frac{1}{2} \right)^2 \times \frac{\pi}{2} - \pi \int_0^{\frac{\pi}{4}} \tan^2 \frac{y}{2} dy$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4} \int_0^{\frac{\pi}{4}} (\sec^2 \frac{y}{2} - 1) dy$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4} \left[2 \tan \frac{y}{2} - y \right]_0^{\frac{\pi}{4}}$$

$$= \frac{\pi^2}{8} - \frac{\pi}{4} \left(2 - \frac{\pi}{2} \right)$$

$$= \frac{\pi^2}{4} - \frac{\pi}{2}$$



$$Q12 \quad V = \pi \int_0^{40} 8(e^{\frac{y}{40}} + 2)^2 dy$$

$$= 64\pi \int_0^{40} e^{\frac{y}{20}} + 4e^{\frac{y}{20}} + 4 dy$$

$$= 64\pi \left[20e^{\frac{y}{20}} + 160e^{\frac{y}{20}} + 4y \right]_0^{40}$$

$$= 64\pi (20e^2 + 160e + 160 - 20 - 160)$$

$$= 1280\pi (e^2 + 8e - 1)$$

$$e^{\frac{y}{40}} = \frac{x-16}{8}$$

$$x = 8e^{\frac{y}{40}} + 16$$

$$Q13 \quad a) \quad V = \pi \int_0^1 16x^4 dx = \pi \left[\frac{16}{5}x^5 \right]_0^1$$

$$= \frac{16\pi}{5}$$

$$b) \quad V = \pi \times 1^2 \times 4 - \pi \int_0^4 \frac{y}{4} dy$$

$$= 4\pi - \pi \left[\frac{y^2}{8} \right]_0^4$$

$$= 2\pi$$

$$c) \quad V = \pi \int_0^4 \frac{y}{4} dy = \pi \left[\frac{y^2}{8} \right]_0^4$$

$$= 2\pi$$

$$\begin{aligned}
 d) \quad V &= \pi \times 4^2 \times 1 - \pi \int_0^1 16x^4 dx \\
 &= 16\pi - \pi \times \frac{16}{5} \\
 &= \frac{64\pi}{5}
 \end{aligned}$$

P1. a) To find volume of region when curve is rotated about y-axis

$$V = \pi \int x^2 dy$$

$$\text{now } x-4 = \pm \sqrt{4-y^2}$$

So two sections are

$$x = 4 + \sqrt{4-y^2} \quad \text{and} \quad x = 4 - \sqrt{4-y^2}$$

$$\begin{aligned}
 b) \quad V &= \pi \int_{-2}^2 (4 + \sqrt{4-y^2})^2 - (4 - \sqrt{4-y^2})^2 dy \\
 &= 2\pi \int_0^2 16 + 8\sqrt{4-y^2} + (4-y^2) - (16 - 8\sqrt{4-y^2} + (4-y^2)) dy \\
 &= 32\pi \int_0^2 \sqrt{4-y^2} dy \\
 c) \quad &= 32\pi \times \frac{1}{4} \pi \times 2^2 \\
 &= 32\pi^2
 \end{aligned}$$

P2. a) $V = \pi \int_4^5 (25-y^2) dy$ Eq. of circle $x^2 + y^2 = 25$
 $= \pi \left[25y - \frac{y^3}{3} \right]_4^5$
 $= \pi \left(125 - \frac{125}{3} - \left(100 - \frac{64}{3} \right) \right)$
 $= \frac{14\pi}{3}$

$$\begin{aligned}
 b) \quad V &= \pi \int_{-5}^4 (25-y^2) dy \\
 &= \pi \left[25y - \frac{y^3}{3} \right]_{-5}^4 \\
 &= \pi \left(100 - \frac{64}{3} - \left(-125 + \frac{125}{3} \right) \right) \\
 &= 162\pi
 \end{aligned}$$

$$\begin{aligned}
 c) \quad V_{\text{sphere}} &= \frac{4}{3} \pi \times 5^3 = \frac{500\pi}{3} \\
 \text{Sum of answers in a) and b)} \\
 \frac{14\pi}{3} + 162\pi &= \frac{500\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 P3 \quad a) \quad V &= \pi \int_{r-h}^r (r^2 - y^2) dy \\
 &= \pi \left[r^2 y - \frac{y^3}{3} \right]_{r-h}^r \\
 &= \pi \left\{ r^3 - \frac{r^3}{3} - \left(r^2(r-h) - \frac{(r-h)^3}{3} \right) \right\} \\
 &= \pi \left\{ r^3 - \frac{r^3}{3} - r^3 + r^2 h + \frac{r^3 - 3r^2 h + 3r h^2 - h^3}{3} \right\} \\
 &= \frac{\pi}{3} (3r h^2 - h^3)
 \end{aligned}$$

$$b) \quad h = r$$

50

Q1 a) $\frac{x}{2} = \sqrt{x}$

$$x^2 = 4x$$

$$\begin{cases} x=0 \\ y=0 \end{cases} \quad \begin{cases} x=4 \\ y=2 \end{cases}$$

b) $A = \int_0^4 \sqrt{x} - \frac{x}{2} dx$
 $= \left[\frac{2}{3} x^{3/2} - \frac{x^2}{4} \right]_0^4 = \frac{2}{3} \cdot 8 - 4$
 $= \frac{4}{3}$

c) i) $V = \pi \int_0^4 x - \frac{x^2}{4} dx$
 $= \pi \left[\frac{x^2}{2} - \frac{x^3}{12} \right]_0^4 = \pi \left(8 - \frac{16}{3} \right)$
 $= \frac{8\pi}{3}$

ii) $V = \pi \int_0^2 4y^2 - y^4 dy$
 $= \pi \left[\frac{4}{3} y^3 - \frac{y^5}{5} \right]_0^2$
 $= \frac{64\pi}{15}$

Q2 a) $x^2 = 8\sqrt{x}$
 $x^4 = 64x$
 $x(x^3 - 64) = 0$
 $\begin{cases} x=0 \\ y=0 \end{cases} \quad \begin{cases} x=4 \\ y=16 \end{cases}$

b) $A = \int_0^4 8\sqrt{x} - x^2 dx$
 $= \left[\frac{16}{3} x^{3/2} - \frac{x^3}{3} \right]_0^4 = \frac{16}{3} \cdot 8 - \frac{64}{3}$
 $= \frac{64}{3}$

c) i) $V = \pi \int_0^4 64x - x^4 dx$
 $= \pi \left[32x^2 - \frac{x^5}{5} \right]_0^4 = \pi \left(32 \cdot 16 - \frac{4^5}{5} \right)$
 $= 307.2\pi = \frac{1536\pi}{5}$

ii) $V = \pi \int_0^{16} y - \left(\frac{y}{8}\right)^4 dy$
 $= \pi \left[\frac{y^2}{2} - \frac{y^5}{5 \cdot 8^4} \right]_0^{16}$
 $= 76.8\pi = \frac{384\pi}{5}$

Q3 a) $x^2 = 27 - 2x^2$
 $3x^2 = 27$
 $\begin{cases} x=3 \\ y=9 \end{cases} \quad \begin{cases} x=-3 \\ y=9 \end{cases}$

b) $A = 2 \int_0^3 27 - 2x^2 - x^2 dx$
 $= 2 \left[27x - x^3 \right]_0^3$
 $= 108$

c) i) $V = 2\pi \int_0^3 (27 - 2x^2)^2 - (x^2)^2 dx$
 $= 2\pi \int_0^3 27^2 - 108x^2 + 3x^4 dx$
 $= 2\pi \left[729x - 36x^3 + \frac{3}{5}x^5 \right]_0^3$
 $= 2721.6\pi$

ii) $V = \pi \int_0^9 y dy + \pi \int_9^{27} \frac{27-y}{2} dy$
 $= \pi \left[\frac{y^2}{2} \right]_0^9 + \frac{\pi}{2} \left[27y - \frac{y^2}{2} \right]_9^{27}$
 $= \frac{81\pi}{2} + \frac{\pi}{2} \left(\frac{27^2}{2} - \left(\frac{405}{2} \right) \right)$
 $= 121.5\pi$

Q4

Q5 a) $A = 2 \int_0^1 x - x^3 dx$
 $= 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1$
 $= \frac{1}{2}$

b) i) $V = 2\pi \int_0^1 x^2 - x^6 dx$
 $= 2\pi \left[\frac{x^3}{3} - \frac{x^7}{7} \right]_0^1$
 $= \frac{8\pi}{21}$

$$\begin{aligned} \text{ii) } V &= 2\pi \int_0^1 y^{2/3} - y^2 dy \\ &= 2\pi \left[\frac{3}{5} y^{5/3} - \frac{y^3}{3} \right]_0^1 \\ &= \frac{8\pi}{15} \end{aligned}$$

$$\text{c) } A = 2 \times \frac{1}{2} = 1$$

$$\begin{aligned} \text{d) i) } V &= 2\pi \int_0^1 x^{2/3} - x^6 dx \\ &= 2\pi \left[\frac{3}{5} x^{5/3} - \frac{x^7}{7} \right]_0^1 \\ &= \frac{32\pi}{35} \end{aligned}$$

$$\begin{aligned} \text{ii) } V &= 2\pi \int_0^1 y^{2/3} - y^6 dy \\ &= \frac{32\pi}{35} \end{aligned}$$

$$\text{Q6 a) } 2\sin x = \operatorname{cosec} x$$

$$2\sin^2 x = 1$$

$$\sin x = \pm \frac{1}{2} \quad \text{neg. values not valid}$$

$$x = \frac{\pi}{4}, \frac{3\pi}{4}$$

$$\begin{aligned} \text{b) } V &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 4\sin^2 x - \operatorname{cosec}^2 x dx \\ &= \pi \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 2 - 2\cos 2x - \operatorname{cosec}^2 x dx \\ &= \pi \left[2x - \sin 2x + \cot x \right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \\ &= \pi \left[\frac{3\pi}{2} - 1 + 1 - \frac{\pi}{2} + 1 - 1 \right] \\ &= \pi^2 \end{aligned}$$

$$\begin{aligned} \text{Q7 } V &= \pi \int_0^{\frac{\pi}{2}} \sec^2 \frac{x}{2} - \tan^2 \frac{x}{2} dx \\ &= \pi \int_0^{\frac{\pi}{2}} 1 dx \\ &= \pi \left[x \right]_0^{\frac{\pi}{2}} \\ &= \frac{\pi^2}{2} \end{aligned}$$

$$\text{Q8 a) Point (1,1) satisfies both equations}$$

$$\begin{aligned} \text{b) } A &= \int_0^1 x^3 dx + \frac{1}{2} \times 1 \times 1 \\ &= \left[\frac{x^4}{4} \right]_0^1 + \frac{1}{2} \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{c) } A &= \int_0^1 y^{1/3} dy + \frac{1}{2} \times 1 \times 1 \\ &= \left[\frac{3}{4} y^{4/3} \right]_0^1 + \frac{1}{2} \\ &= \frac{5}{4} \end{aligned}$$

$$\begin{aligned} \text{d) } V &= \pi \int_0^1 x^6 dx + \pi \int_1^2 4 - 4x + x^2 dx \\ &= \pi \left[\frac{x^7}{7} \right]_0^1 + \pi \left[4x - 2x^2 + \frac{x^3}{3} \right]_1^2 \\ &= \pi \left(\frac{1}{7} + 8 - 8 + \frac{8}{3} - 4 + 2 - \frac{1}{3} \right) \\ &= \frac{10\pi}{21} \end{aligned}$$

$$\begin{aligned} \text{e) } V &= \pi \int_0^1 y^{2/3} dy + \pi \int_1^2 4 - 4y + y^2 dy \\ &= \pi \left[\frac{3}{5} y^{5/3} \right]_0^1 + \pi \left[4y - 2y^2 + \frac{y^3}{3} \right]_1^2 \\ &= \pi \left(\frac{3}{5} + 8 - 8 + \frac{8}{3} - 4 + 2 - \frac{1}{3} \right) \\ &= \frac{14\pi}{15} \end{aligned}$$

$$\begin{aligned} \text{Q9 a) } V &= \pi \int_0^{\frac{\pi}{2}} 16 \cos^2 \frac{x}{2} - 16 \sin^2 \frac{x}{2} dx \\ &= 16\pi \int_0^{\frac{\pi}{2}} \cos x dx \\ &= 16\pi \left[\sin x \right]_0^{\frac{\pi}{2}} \\ &= 16\pi \end{aligned}$$

$$\begin{aligned} \text{b) } V &= \pi \int_0^{\frac{\pi}{2}} 16 \sin^2 \frac{x}{2} + \pi \int_{\frac{\pi}{2}}^{\pi} 16 \cos^2 \frac{x}{2} dx \\ &= \frac{16\pi}{2} \int_0^{\frac{\pi}{2}} 1 - \cos x + \frac{16\pi}{2} \int_{\frac{\pi}{2}}^{\pi} 1 + \cos x dx \\ &= 8\pi \left[\frac{x}{2} - 1 + \pi - \frac{\pi}{2} - 1 \right] \\ &= 8\pi (\pi - 2) \end{aligned}$$

Q10 a) i) $V = \pi \int_0^3 \frac{16x^2}{9} dx + \pi \int_3^5 25 - x^2 dx$

ii) $V = \pi \left[\frac{16x^3}{27} \right]_0^3 + \pi \left[25x - \frac{x^3}{3} \right]_3^5$
 $= 16\pi + \frac{52\pi}{3}$
 $= \frac{100\pi}{3}$

$$25 - x^2 = \frac{16x^2}{9}$$

$$225 - 9x^2 = 16x^2$$

$$25x^2 = 225$$

$x = 3$ in this diagram

b) i) $V = \pi \int_0^4 25 - y^2 - \frac{9y^2}{16} dy$
 $= \pi \int_0^4 25 - \frac{25y^2}{16} dy$

ii) $V = \pi \left[25y - \frac{25}{16 \times 3} y^3 \right]_0^4$
 $= \frac{200\pi}{3}$

c) i) $V = \pi \int_0^3 25 - x^2 - \frac{16x^2}{9} dx$
 $= \pi \int_0^3 25 - \frac{25x^2}{9} dx$

ii) $V = \pi \left[25x - \frac{25}{9 \times 3} x^3 \right]_0^3$
 $= 50\pi$

d) i) $V = \pi \int_0^4 \frac{9y^2}{16} dy + \pi \int_4^5 25 - y^2 dy$

ii) $V = \pi \left[\frac{9}{16 \times 3} y^3 \right]_0^4 + \pi \left[25y - \frac{y^3}{3} \right]_4^5$
 $= 12\pi + \frac{14\pi}{3}$
 $= \frac{50\pi}{3}$

P1 a) $V = \pi \int_0^1 (2x - x^2)^2 - x^4 dx$
 $= \pi \int_0^1 4x^2 - 4x^3 dx$
 $= \pi \left[\frac{4x^3}{3} - x^4 \right]_0^1$
 $= \frac{\pi}{3}$

$$1 - (x-1)^2 = 2x - x^2$$

b) $V = \pi \int_0^1 y - (1 - \sqrt{1-y})^2 dy$
 $= \pi \left[\frac{y^2}{2} - 2y + \frac{2}{3}(1-y)^{3/2} + \frac{y^2}{2} \right]_0^1$
 $= \pi \left(\frac{1}{2} - 2 + \frac{1}{2} + \frac{4}{3} \right)$
 $= \frac{\pi}{3}$

$$(x-1)^2 = 1-y$$

$$x-1 = \pm \sqrt{1-y}$$

$$x = 1 \pm \sqrt{1-y}$$

$= 1 - \sqrt{1-y}$ this branch is relevant in this case.

P2 a) $V = \pi \int_0^1 (1-x)^2 - (x-1)^{2n} dx$
 $= \pi \left[\frac{(1-x)^3}{3} - \frac{(x-1)^{2n+1}}{2n+1} \right]_0^1$
 $= \pi \left(\frac{1}{3} - \frac{1}{2n+1} \right)$

$$(1 - \sqrt{1-y})^2$$

$$= 1 - 2\sqrt{1-y} + 1-y$$

$$= 2 - 2\sqrt{1-y} - y$$

b) as $n \rightarrow \infty$
 $\frac{1}{2n+1} \rightarrow 0$
 $\therefore V \rightarrow \frac{\pi}{3}$

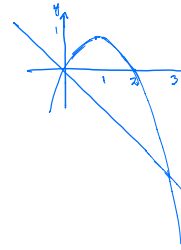
c) see answer

Review

Q1 a) $A = 2 \int_0^9 \sqrt{9-x} \, dx = -2 \cdot \frac{2}{3} [(9-x)^{3/2}]_0^9$
 $= 36$

b) $A = \int_0^6 9 - \frac{y^2}{4} \, dy = [9y - \frac{y^3}{12}]_0^6$
 $= 36$

$(\frac{y}{2})^2 = 9-x$
 $x = 9 - \frac{y^2}{4}$



Q2 $A = \int_0^3 2x - x^2 + x \, dx$
 $= \int_0^3 [\frac{3x^2}{2} - \frac{x^3}{3}] \, dx$
 $= 4.5$

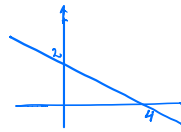
Q3 $A = 2 \int_0^2 \frac{4}{4+x^2} \, dx = \frac{2 \cdot 4}{2} [\tan^{-1} \frac{x}{2}]_0^2$
 $= \pi$

Q4 $V = \pi \int y^2 \, dx = \pi \int_0^3 4x^2 \, dx$
 $= \pi [\frac{4x^3}{3}]_0^3$
 $= 36\pi$

Q5 a) $V = \pi \int_0^9 x \, dx = [\frac{\pi x^2}{2}]_0^9$
 $= \frac{81\pi}{2}$

b) $V = 2\pi \int_0^3 (9^2 - y^4) \, dy = 2\pi [81y - \frac{y^5}{5}]_0^3$
 $= \frac{1944\pi}{5} = 388.8\pi$

Q6 a) $V = \frac{1}{3} \pi \times 2^2 \times 4$ or $V = \pi \int_0^4 (\frac{4-x}{2})^2 \, dx$
 $= \frac{16\pi}{3}$
 $= \frac{\pi}{4} [\frac{(4-x)^3}{3}]_0^4$
 $= \frac{\pi}{4} \times \frac{64}{3}$
 $= \frac{16\pi}{3}$



b) $V = \frac{1}{3} \pi \times 4^2 \times 2$ or $V = \pi \int_0^2 (4-2y)^2 \, dy$
 $= \frac{32\pi}{3}$
 $= \pi [\frac{(4-2y)^3}{3 \times -2}]_0^2$
 $= \frac{32\pi}{3}$

Q7 a) $V = \pi \times 1^2 \times 1 - \pi \int_0^1 x \, dx$
 $= \pi - \pi [\frac{x^2}{2}]_0^1$
 $= \frac{\pi}{2}$

b) $V = \pi \times 1^2 \times 1 - \pi \int_0^1 x^4 \, dx$
 $= \pi - [\frac{\pi x^5}{5}]_0^1$
 $= \frac{4\pi}{5}$

c) $V = \pi \times 1^2 \times 1 - \int_0^1 y^4 \, dy$
 $= \frac{4\pi}{5}$

d) $V = \pi \times 1^2 \times 1 - \pi \int_0^1 y \, dy$
 $= \frac{\pi}{2}$

Q8 a) $V = \pi \int_0^1 x - x^4 \, dx$
 $= \pi [\frac{x^2}{2} - \frac{x^5}{5}]_0^1$
 $= \frac{3\pi}{10}$

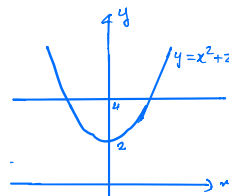
b) $V = \pi \int_0^1 y - y^4 \, dy$
 $= \frac{3\pi}{10}$

Q9 a) $A = \int_1^6 \frac{6}{x} \, dx = [6 \ln x]_1^6$
 $= 6 \ln 6$

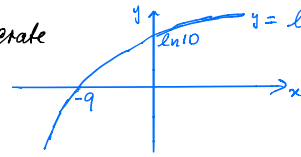
b) $V = \pi \int_1^6 \frac{36}{x^2} \, dx = [-\frac{36}{x}]_1^6$
 $= 30\pi$

Q10 a) $V = \pi \int_2^4 y - 2 \, dy = \pi [\frac{y^2}{2} - 2y]_2^4$
 $= 2\pi$

b) $V = 2\pi \int_0^{\sqrt{2}} 16 - (x^2+2)^2 \, dx$
 $= 2\pi \int_0^{\sqrt{2}} 12 - x^4 - 4x^2 \, dx$
 $= 2\pi [12x - \frac{x^5}{5} - \frac{4}{3}x^3]_0^{\sqrt{2}}$
 $= 2\pi (12\sqrt{2} - \frac{4\sqrt{2}}{5} - \frac{8\sqrt{2}}{3})$
 $= 17\frac{1}{15}\sqrt{2}\pi = \frac{256\sqrt{2}\pi}{15}$



Q11 a) $A = \int_0^{\ln 10} x \, dy$ Note: we can't integrate a log.
 $= - \int_0^{\ln 10} e^y - 10 \, dy$
 $= - [e^y - 10y]_0^{\ln 10}$
 $= - (10 - 10 \ln 10 - 1)$
 $= 10 \ln 10 - 9$



b) $V = \pi \int_0^{\ln 10} e^{2y} - 20e^y + 100 \, dy$
 $= \pi [\frac{1}{2}e^{2y} - 20e^y + 100y]_0^{\ln 10}$
 $= \pi (50 - 20 \times 10 + 100 \ln 10 - \frac{1}{2} + 20)$
 $= (100 \ln 10 - 130.5) \pi$

Q12 a) $A = 2 \int_0^{\frac{\pi}{6}} 2 \cos 3x \, dx$
 $= \frac{4}{3} [\sin 3x]_0^{\frac{\pi}{6}}$
 $= \frac{4}{3}$

b) $V = 2\pi \int_0^{\frac{\pi}{6}} 4 \cos^2 3x \, dx$
 $= 4\pi \int_0^{\frac{\pi}{6}} \cos 6x + 1 \, dx$
 $= 4\pi [\frac{1}{6} \sin 6x + x]_0^{\frac{\pi}{6}}$
 $= 4\pi (\frac{\pi}{6})$
 $= \frac{2\pi^2}{3}$

Q13 a) $A = 4 \int_0^{12} \frac{1}{\sqrt{x+4}} \, dx$
 $= 4 \times 2 [(x+4)^{\frac{1}{2}}]_0^{12}$
 $= 8(4 - 2)$
 $= 16$

b) $V = \pi \int_0^{12} \frac{16}{x+4} \, dx$
 $= 16\pi [\ln(x+4)]_0^{12}$
 $= 16\pi (\ln 16 - \ln 4)$
 $= 32\pi \ln 2$

Q14 $V = \pi \int_0^{\pi} \sin^2 x \, dx$
 $= \frac{\pi}{2} \int_0^{\pi} 1 - \cos 2x \, dx$
 $= \frac{\pi}{2} [x - \frac{1}{2} \sin 2x]_0^{\pi}$
 $= \frac{\pi^2}{2}$

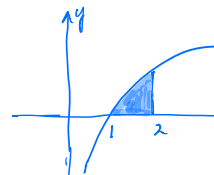
Q15 $V = 16\pi \int_0^{\ln 50} e^{-x} \, dx$
 $= 16\pi [-e^{-x}]_0^{\ln 50}$
 $= 16\pi (1 - \frac{1}{50})$
 $= \frac{392\pi}{5}$

Q16 $V = \frac{2\pi}{100} \int_0^1 e^{8x} + 2 + e^{-8x} \, dx$
 $= \frac{\pi}{50} [\frac{1}{8}e^{8x} + 2x - \frac{1}{8}e^{-8x}]_0^1$
 $= \frac{\pi}{50} (\frac{1}{8}e^8 + 2 - \frac{1}{8}e^{-8} - \frac{1}{8} + \frac{1}{8})$
 $= \frac{\pi}{50} (2 + \frac{e^8}{8} - \frac{e^{-8}}{8})$

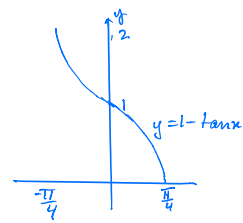
Q17 a) $V = 2\pi \int_0^{\frac{\pi}{6}} \cos 3x \, dx$
 $= 2\pi [\frac{1}{3} \sin 3x]_0^{\frac{\pi}{6}}$
 $= \frac{2\pi}{3}$

b) $V = \pi \int_0^{4\pi} \sin \frac{x}{4} \, dx$
 $= 4\pi [-\cos \frac{x}{4}]_0^{4\pi}$
 $= 4\pi (1 + 1)$
 $= 8\pi$

Q18 $V = \pi \int_0^{\ln 2} 4 - e^{2y} \, dy$
 $= \pi [4y - \frac{1}{2}e^{2y}]_0^{\ln 2}$
 $= \pi (4 \ln 2 - 2 + \frac{1}{2})$
 $= \pi (4 \ln 2 - 1.5)$



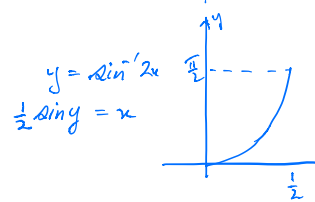
$$\begin{aligned}
 Q19 \quad a) \quad A &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 - \tan x \, dx \\
 &= \left[x + \ln \cos x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
 &= \frac{\pi}{4} + \ln \frac{1}{\sqrt{2}} + \frac{\pi}{4} - \ln \frac{1}{\sqrt{2}} \\
 &= \frac{\pi}{2}
 \end{aligned}$$



$$\begin{aligned}
 b) \quad V &= \pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 1 - 2 \tan x + \tan^2 x \, dx \\
 &= \pi \left[x - 2 \ln \cos x + \tan x - x \right]_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \\
 &= \pi (1 - 2 \ln \frac{1}{\sqrt{2}} + 1 + 2 \ln \frac{1}{\sqrt{2}}) \\
 &= 2\pi
 \end{aligned}$$

$$\begin{aligned}
 Q20 \quad V &= \pi \int_0^5 \frac{1}{25+x^2} \, dx \\
 &= \frac{\pi}{5} \left[\tan^{-1} \frac{x}{5} \right]_0^5 \\
 &= \frac{\pi^2}{20}
 \end{aligned}$$

$$\begin{aligned}
 Q21 \quad V &= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \sin^2 y \, dy \\
 &= \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{1 - \cos 2y}{2} \, dy \\
 &= \frac{\pi}{8} \left[y - \frac{1}{2} \sin 2y \right]_0^{\frac{\pi}{2}} \\
 &= \frac{\pi}{8} \left(\frac{\pi}{2} \right) \\
 &= \frac{\pi^2}{16}
 \end{aligned}$$



$$\begin{aligned}
 Q22 \quad a) \quad V &= \pi \int_0^{\frac{\pi}{8}} 9 \cos^2 2x - 9 \sin^2 2x \, dx \\
 &= 9\pi \int_0^{\frac{\pi}{8}} \cos 4x \, dx \\
 &= \frac{9\pi}{4} \left[\sin 4x \right]_0^{\frac{\pi}{8}} \\
 &= \frac{9\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 b) \quad V &= 9\pi \int_0^{\frac{\pi}{8}} \sin^2 2x \, dx + 9\pi \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cos^2 2x \, dx \\
 &= \frac{9\pi}{2} \int_0^{\frac{\pi}{8}} 1 - \cos 4x \, dx + \frac{9\pi}{2} \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} 1 + \cos 4x \, dx \\
 &= \frac{9\pi}{2} \left[x - \frac{1}{4} \sin 4x \right]_0^{\frac{\pi}{8}} + \left[x + \frac{1}{4} \sin 4x \right]_{\frac{\pi}{8}}^{\frac{\pi}{4}} \\
 &= \frac{9\pi}{2} \left(\frac{\pi}{8} - \frac{1}{4} + \frac{\pi}{4} - \frac{\pi}{8} - \frac{1}{4} \right) \\
 &= \frac{9\pi}{2} (\pi - 2)
 \end{aligned}$$

$$\begin{aligned}
 Q23 \quad V &= \pi \int_0^1 4 \cos^2 \frac{\pi x}{3} - x^2 \, dx \\
 &= \pi \int_0^1 2 \left(\cos \frac{2\pi x}{3} + 1 \right) - x^2 \, dx \\
 &= \pi \left[\frac{2x}{2\pi} \sin \frac{2\pi x}{3} + 2x - \frac{x^3}{3} \right]_0^1 \\
 &= \pi \left(\frac{2}{\pi} \times \frac{\sqrt{3}}{2} + 2 - \frac{1}{3} \right) \\
 &= \frac{5\pi}{3} + \frac{3\sqrt{3}}{2}
 \end{aligned}$$

$$\begin{aligned}
 Q24 \quad V &= \pi \int_0^{30} (10e^{\frac{y}{30}} + 15)^2 \, dy \\
 &= 25\pi \int_0^{30} 4e^{\frac{y}{15}} + 12e^{\frac{y}{30}} + 9 \, dy \\
 &= 25\pi \left[60e^{\frac{y}{15}} + 360e^{\frac{y}{30}} + 9y \right]_0^{30} \\
 &= 25\pi [60e^2 + 360e + 270 - 60 - 360] \\
 &= 750\pi [2e^2 + 12e - 5]
 \end{aligned}$$

$$\begin{aligned}
 y &= 30 \ln \left(\frac{x-15}{10} \right) \\
 e^{\frac{y}{30}} &= \frac{x-15}{10} \\
 x &= 10e^{\frac{y}{30}} + 15
 \end{aligned}$$