8A

Q3 a) $x = \log_{10} 10^3 = 3\log_{10} 10 = 3$

b) $x = \log_{10} 10^{-2} = -2\log_{10} 10 = -2$

c) $x = \log_2 2^2 = 2\log_2 2 = 2$

d) $x = \log_4 4^{-1} = -1\log_4 4 = -1$

e) $x = \log_{\frac{1}{2}} \left(\frac{1}{2}\right)^{-4} = -4\log_{\frac{1}{2}} \frac{1}{2} = -4$

f) $x = \log_{12} 12 = 1$

g) $x = \log_{12} 12^0 = 0$

h) $x^2 = 25 \qquad x = 5$

i) $x^3 = 4^3 \qquad x = 4$

j) $x^{-2} = 5^{-2} \qquad x = 5$

k) $x^{\frac{1}{2}} = 3 \qquad x = 9$

l) $x^{3/2} = 3^3 \qquad x^3 = 9^3 \qquad x = 9$

m) $3^{-2} = x \qquad x = \frac{1}{9}$

n) $x = 2^3 = 8$

Q6 a) 4

b) $\log_3 3^4 = 4$

c) $\log_5 5^{\frac{1}{2}} = \frac{1}{2}$

d) $\log_2 2^{5/2} = \frac{5}{2}$

P1 b) Since $a^{\log_a(x)} = x$

$4^{\log_4 125} = 125$

c) $25^{\log_5(4)} = 5^{2\log_5(4)}$

$= 5^{\log_5 16} = 16$

P2 We need the intersection of $x > 0$ and $x - 2 > 0$

Hence domain $x > 2$

8B

Q2 a) $\log_2 \frac{32}{64} = \log_2 \frac{1}{2} = \log_2 2^{-1}$
$= -1$

b) $\log_5 \frac{100}{4} = \log_5 25 = 2\log_5 5$
$= 2$

c) $\frac{1}{2}\log_{10} 10 \times \frac{1}{3}\log_{10} 10 = \frac{1}{6}$

d) $\frac{1}{2}\log_3 3 - (-2)\log_3 3 + 3\log_3 3$
$= \frac{1}{2} + 2 + 3 = 5\frac{1}{2}$

Q3 Use the change of base laws to base 10 then use your calculator

Q4 a) $\doteqdot \log_{10} 10^2 = 2$

b) $\doteqdot \log_{10} 10^{-2} = -2$

c) $\doteqdot \log_3 3^3 = 3$

d) $\doteqdot \log_2 2^5 = 5$

e) $\doteqdot \log_2 2^{-3} = -3$

f) $\doteqdot \log_3 3^5 = 5$

g) $\doteqdot \log_2 2^7 = 7$

h) $\doteqdot \log_3 3^4 = 4$

i) $\doteqdot \log_3 3^{-3} = -3$

Q9 a) $\frac{\log_2 2^5}{\log_2 2^3} = \frac{5}{3}$

b) $\frac{\log_2 2^5}{\log_2 2^{-2}} = -\frac{5}{2}$

c) $\dfrac{\log_3 3^3}{\log_3 3^{-2}} = -\dfrac{3}{2}$

d) $\dfrac{\log_5 5^3}{\log_5 5^2} = \dfrac{3}{2}$

Q10 a) $\log_a b = \dfrac{\log_b b}{\log_b a} = \dfrac{1}{\log_b a}$

b) $\log_{\sqrt{27}} 3 = \dfrac{1}{\log_3 \sqrt{27}} = \dfrac{1}{\log_3 3^{3/2}}$

$= \dfrac{1}{\frac{3}{2}} = \dfrac{2}{3}$

Q11 $LHS = \dfrac{\log b}{\log a} \times \dfrac{\log c}{\log b} \times \dfrac{\log a}{\log c}$

$= 1 = RHS.$

$LHS = \dfrac{\log b^x}{\log a^x} = \dfrac{x \log b}{x \log a}$

$= \dfrac{\log b}{\log a} = \log_a b = RHS.$

P1 a) $\log_5 y = 3\log_5 5 - 2\log_5 x$

$= \log_5 \dfrac{5^3}{x^2}$

$\therefore\ y = \dfrac{125}{x^2}$

b) $x > 0$

P2 $a^2 + b^2 = 7ab$

$(a+b)^2 = 9ab$

a) $\dfrac{(a+b)^2}{9} = ab$

$\dfrac{a+b}{3} = \sqrt{ab}$

b) $\log\left(\dfrac{a+b}{3}\right) = \log\sqrt{ab}$

$= \frac{1}{2}(\log ab) = \frac{1}{2}(\log a + \log b)$

P3 $\log_{10}\left(\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{99}{100}\right) = \log_{10}\frac{1}{100}$

$= \log_{10} 10^{-2}$

$= -2$

P4 $\log a = p \log x$

$\log a = \log x^p$

$\therefore\ a = x^p$

P5

$$LHS = \frac{\log_x a}{\log_x xy} = \frac{\log_x a}{\log_x x + \log_x y}$$

$$= \frac{\log_x a}{1 + \log_x y} = RHS$$

P6

$$LHS = \frac{\log_a b}{\log_a a^x} = \frac{\log_a b}{x \log_a a}$$

$$= \frac{\log_a b}{x} = RHS.$$

P7

a) $\left(c^{\log_c a}\right)^{\log_b c} = a^{\log_b c}$

b) considering the power of c

$$\frac{\log a}{\log c} \times \frac{\log c}{\log b} = \frac{\log a}{\log b} = \log_b a$$

$\therefore$ in a) $LHS = c^{\log_b a}$

$= a^{\log_b c}$ proved in a)

8C

Q4 a) $2 = e^{\ln 2}$

b) $2^x = e^{x\ln 2}$ $\quad a = \ln 2.$

c) $\frac{d}{dx}(2^x) = \frac{d}{dx}(e^{x\ln 2})$

$= \ln 2 \, e^{x\ln 2}$

$= 2^x \ln 2.$

Q5 c) $3^{2x} = 9^x$ $\quad \frac{d}{dx}(9^x) = 9^x \ln 9$

$= 3^{2x}\, 2\ln 3$

d) $4^{\frac{x}{2}} = 2^x$ $\quad \frac{d}{dx}(2^x) = 2^x \ln 2$

Q7 a) $y = e^{2x}$ $\quad y' = 2e^{2x}$ $\quad y'' = 4e^{2x}$

b) $y''' = 2^3 e^{2x}$

$y^{iv} = 2^4 e^{2x}$ ... nth derivative $= 2^n e^{2x}$

Q8 $P = P_0 e^{kt}$

$\frac{dP}{dt} = P_0 k e^{kt}$

$= k(P_0 e^{kt}) = kP$

Q9 $y = Ae^x + Be^{-x}$

$y' = Ae^x - Be^{-x}$

$y'' = Ae^x + Be^{-x}$

$LHS = y'' - y$

$= Ae^x + Be^{-x} - (Ae^x + Be^{-x})$

$= 0 = RHS.$

Q10 $y = Ae^{-2x} + Be^{-3x}$

$y' = -2Ae^{-2x} - 3Be^{-3x}$

$y'' = 4Ae^{-2x} + 9Be^{-3x}$

$LHS = 4Ae^{-2x} + 9Be^{-3x} + 5(-2Ae^{-2x} - 3Be^{-3x}) +$

$+ 6(Ae^{-2x} + Be^{-3x})$

$= 0 = RHS.$

Q11

$$y = e^{kx}$$
$$y' = ke^{kx}$$
$$y'' = k^2e^{kx}$$
$$y'' + 5y' = 6y$$
$$k^2e^{kx} + 5ke^{kx} - 6e^{kx} = 0$$
$$k^2 + 5k - 6 = 0 \qquad e^{kx} \neq 0$$
$$(k+6)(k-1) = 0$$
$$k = -6, 1$$

Q12

a) $LHS = \dfrac{e^x + e^{-x} + e^x - e^{-x}}{2} = \dfrac{2e^x}{2} = RHS$

b) $LHS = \dfrac{e^x + e^{-x} - e^x + e^{-x}}{2} = \dfrac{2e^{-x}}{2} = RHS$

c) $c'(x) = \dfrac{e^x - e^{-x}}{2} = s(x)$

d) $s'(x) = \dfrac{e^x + e^{-x}}{2} = c(x)$

e) $s(2x) = \dfrac{e^{2x} - e^{-2x}}{2} = \dfrac{(e^x + e^{-x})(e^x - e^{-x})}{2}$

$$= 2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)$$
$$= 2\, c(x)\, s(x)$$

b) $LH = c(2x) = \dfrac{e^{2x} + e^{-2x}}{2}$

$RHS = s^2(x) + c^2(x)$

$$= \frac{e^{2x} - 2 + e^{-2x} + e^{2x} + 2 + e^{-2x}}{4}$$
$$= \frac{e^{2x} + e^{-2x}}{2} = LHS.$$

g) $LHS = \dfrac{1}{4}\left(e^{2x} + 2 + e^{-2x} - (e^{2x} - 2 + e^{-2x})\right)$

$$= \frac{1}{4}(4) = 1 = RHS.$$

h) $LHS = \frac{1}{2}(e^{2x} + e^{-2x})$

$RHS = 2 \times \frac{1}{4}(e^{2x} - 2 + e^{-2x}) + 1$

$= \frac{1}{2}(e^{2x} + e^{-2x}) - \frac{1}{2}(2) + 1$

$= LHS.$

i) $LHS = s(x+y)$

$= \frac{1}{2}(e^{x+y} - e^{-(x+y)})$

$RHS = s(x)c(y) + c(x)s(y)$

$= \frac{1}{4}(e^{x} - e^{-x})(e^{y} + e^{-y}) + \frac{1}{4}(e^{x} + e^{-x})(e^{y} - e^{-y})$

$= \frac{1}{4}(2e^{x+y} - 2e^{-(x+y)})$ middle terms cancel

$= LHS.$

j) $LHS = c(x+y)$

$= \frac{1}{2}(e^{x+y} + e^{-(x+y)})$

$RHS = \frac{1}{4}\{(e^{x} + e^{-x})(e^{y} + e^{-y}) + (e^{x} - e^{-x})(e^{y} - e^{-y})\}$

$= \frac{1}{2}(e^{x+y} + e^{-(x+y)})$ Middle terms cancel

P1 a) $e^{x \ln a} = e^{\ln a^{x}} = a^{x}$

b) $\frac{d(a^{x})}{dx} = \frac{d e^{x \ln a}}{dx} = \ln a \, e^{x \ln a} = a^{x} \ln a$

P2

$y = ae^{\alpha x} + be^{\beta x}$

$y' = a\alpha e^{\alpha x} + b\beta e^{\beta x}$

$y'' = a\alpha^{2} e^{\alpha x} + b\beta^{2} e^{\beta x}$

$LHS = ay'' + by' + cy$

$= a^{2}\alpha^{2}e^{\alpha x} + ab\beta^{2}e^{\beta x} + ab\alpha e^{\alpha x} + b^{2}\beta e^{\beta x}$
$+ ace^{\alpha x} + bce^{\beta x}$

$= e^{\alpha x}a(a\alpha^{2} + b\alpha + c) + e^{\beta x}b(a\beta^{2} + b\beta + c)$

now since α and β are roots of quadratic equation $ax^{2} + bx + c = 0$

$LHS = e^{\alpha x} a \times 0 + e^{\beta x} b \times 0$

$= 0 = RHS.$

8D

Q2 a) $2^{2x-4} = 2^{3x-4}$

$2x-4 = 3x-4$

$x = 0$

b) $4^{x+2} = 4^{2x+2}$

$x+2 = 2x+2$

$x = 0$

c) $5^{2x+2} = 5^{6x-3}$

$2x+2 = 6x-3$

$4x = 5$

$x = \frac{5}{4}$

Q3 a) $2^{x+y} = 2^2 \Rightarrow x+y = 2$

$3^{x-y} = 3^3 \quad x-y = 3$

$\therefore \quad x = \frac{5}{2} \quad y = -\frac{1}{2}$

b) $5^{2x+y} = 5^0 \Rightarrow 2x+y = 0$

$2^{x-2y} = 2^5 \quad x-2y = 5$

$x - 2(-2x) = 5$

$x = 1 \quad y = -2$

c) $4^{2x-y} = 4^3 \Rightarrow 2x-y = 3$

$2^{3x+y} = 2^2 \quad 3x+y = 2$

$x = 1 \quad y = -1$

Q4 a) Let $7^x = u$

$u^2 - 6u - 7 = 0$

$(u-7)(u+1) = 0$

$u = 7 \quad u = -1$

no soln.

$7^x = 7$

$x = 1$

b) Let $3^x = u$

$u^2 - 2u - 3 = 0$

$(u-3)(u+1) = 0$

$u = 3 \quad u = -1$

$3^x = 3 \quad 3^x = -1$

$x = 1 \quad$ no solution

c) Let $2^x = u$

$u^2 + 2u - 8 = 0$

$(u+4)(u-2) = 0$

$u = -4$ $\quad$ $u = 2$

no soln $\quad$ $2^x = 2$

$x = 1$

Q5 a) $2x+1 = 3^2$

$x = 4$

b) $2 = x^{\frac{1}{2}}$

$x = 4$

c) $x^4 = 3^4$

$x = 3$

d) $x = 9^{3/2}$

$x = 27$

e) $x^{-3} = 4^{-3}$

$x = 4$

f) $x = 4^{\frac{1}{4}}$

$x = \sqrt{2}$

Q6 a) $\log_2 (x-1)(x+1) = 3$

$x^2 - 1 = 8$

$x^2 = 9$

$x = 3$ (Note $x \neq -3$

b) $\log_{10} \frac{x+1}{x-8} = 1$

$\frac{x+1}{x-8} = 10$

$x+1 = 10x - 80$

$9x = 81$

$x = 9$

c) $x-1 = (x+1)^2$

$x^2 + x = 0$

$x(x+1) = 0$

$x = 0$ or $x = -1$

neither solution satisfies the domain

$\therefore$ no solution

d) Let $\ln x = u$

$$u^2 - 2u + 1 = 0$$
$$(u-1)^2 = 0$$
$$u = 1$$
$$\ln x = 1$$
$$x = e$$

e) Let $\log_3 x = u$

$$u^2 - 5u + 6 = 0$$
$$(u-3)(u-2) = 0$$
$$u = 3 \qquad u = 2$$
$$\log_3 x = 3 \qquad \log_3 x = 2$$
$$x = 27 \qquad x = 9$$

f) $(2x+4)(x-1) = (x+1)^2$

$$2x^2 + 2x - 4 = x^2 + 2x + 1$$
$$x^2 = 5$$

$x = \sqrt{5}$ Note $x = -\sqrt{5}$ is not a valid solution

Q7 $x \ln 2 = \ln 5$

$$x = \frac{\ln 5}{\ln 2}$$

$$x = 2.32$$

Q8 a) $x \ln 2 = \ln 10$

$$x = \frac{\ln 10}{\ln 2} = 3.32$$

b) $2x = \frac{\ln 8}{\ln 5}$

$$x = 0.65$$

c) no solution

d) $3x - 2 = \frac{\ln 15}{\ln 3}$

$$x = 1.49$$

e) $e^{-5x} = \frac{2}{3}$

$$-5x = -0.405465$$
$$x = 0.08$$

f) $e^x(3e^x - 2) = 0$

$e^x = 0$ or $e^x = \frac{2}{3}$

no soln $\quad x = -0.41$

g) $e^x \times e^{\ln 4} = 3e^x + 2$

$4e^x = 3e^x + 2$

$e^x = 2$

$x = 0.69$

h) $e^x(e^x + 1) = 0$

$e^x = 0 \quad e^x = -1$

no solution

Q9 a) Let $2^x = u$

$u^2 - 6u + 5 = 0$

$(u-5)(u-1) = 0$

$u = 5$ or $u = 1$

$2^x = 5 \quad 2^x = 1$

$x = \frac{\ln 5}{\ln 2} \quad x = 0$

$= 2.32$

b) Let $3^x = u$

$u^2 - 3u + 2 = 0$

$(u-2)(u-1) = 0$

$3^x = 2 \quad 3^x = 1$

$x = \frac{\ln 2}{\ln 3} \quad x = 0$

$= 0.63$

c) Let $5^x = u$

$u^2 - 5u - 6 = 0$

$(u-6)(u+1) = 0$

$u = 6 \quad u = -1$

$5^x = 6 \quad$ no soln.

$x = \frac{\ln 6}{\ln 5}$

$= 1.11$

Q10

a) $V_2 = 2000(1.05)^2 = \$2205$

b) $V_t = 2000(1.05)^t$

c) $4000 = 2000(1.05)^t$

$(1.05)^t = 2$

$t = \dfrac{\ln 2}{\ln 1.05} = 14.2$

15 years

d) Yes he is correct :

If value is P and doubled it will be 2P

so $2P = P(1.05)^t$

$(1.05)^t = 2$ as before

yielding $t = 15$ years.

Q11

a) $v = 10(1 - e^{-2}) = 8.65$

b) $5 = 10(1 - e^{-t})$

$e^{-t} = \frac{1}{2}$

$t = \ln 2 = 0.69$

c) The speed will approach 10 as t gets large but will never actually reach it.

Q12

a) $C(1) = 9(e^{-0.2} - e^{-0.4})$

$= 1.3357\ \mu g\ mL^{-1}$

b) $x = e^{-0.2t}$ then $e^{-0.4t} = (e^{-0.2t})^2$

$= x^2$

$\therefore 2 = 9(x - x^2)$

$9x^2 - 9x + 2 = 0$

c) $(3x-2)(3x-1) = 0$

$x = \frac{2}{3}$ $\quad x = \frac{1}{3}$

$\frac{2}{3} = e^{-0.2t}$ $\quad \frac{1}{3} = e^{-0.2t}$

$t = 2.027$ $\quad t = 5.493$

So t is just over 2 hours

P1

a) $3^1 \times 3^{2x+2} = 3^{3x-6}$

$2x+3 = 3x-6$

$x = 9$

b) $2^{4(2-x)} = 2^{3x/2}$

$8 - 4x = \frac{3x}{2}$

$16 - 8x = 3x$

$11x = 16$

$x = \frac{16}{11}$

c) $\left(\frac{2}{3}\right)^{3x} = \left(\frac{3}{2}\right)^{4-2x}$

$4 - 2x = -3x$

$x = -4$

P2

a) $v = b(1 - e^{-kt})$

$10 = b(1 - e^{-k})$

$15 = b(1 - e^{-2k})$

$\frac{3}{2} = \frac{1 - e^{-2k}}{1 - e^{-k}}$ Dividing

$3 - 3e^{-k} = 2 - 2e^{-2k}$

$2e^{-2k} - 3e^{-k} + 1 = 0$

b) $2p^2 - 3p + 1 = 0$

$(2p-1)(p-1) = 0$

$p = \frac{1}{2}$ $p = 1$

c) $e^{-k} = \frac{1}{2}$ $e^{-k} = 1$

$k = \ln 2$ $k \neq 0$

d) $10 = b(1 - e^{-\ln 2}) = b(1 - \frac{1}{2})$

$\therefore b = 20$

e) $v = 20(1 - e^{-t\ln 2})$

as $t \to \infty$ $v \to 20$

f) 20m/s. is the terminal vel

P3

$\frac{\log_2 x}{\log_2 8} + \frac{\log_2 x}{\log_2 4} = 5$

$\frac{\log_2 x}{3} + \frac{\log_2 x}{2} = 5$

$5\log_2 x = 30$

$\log_2 x = 6$

$x = 64$

8E

P1

$$2x\ln 2 = (x-1)\ln 5$$
$$x(2\ln 2 - \ln 5) = -\ln 5$$
$$x = \frac{\ln 5}{\ln 5 - \ln 4} = \frac{\ln 5}{\ln \frac{5}{4}}$$

P2

Let $2^x = u$

$$u^2 + u - 6 = 0$$
$$(u+3)(u-2) = 0$$
$u = -3$ or $u = 2$

no soln $\quad 2^x = 2$

$$x = 1$$

P3

a) $c = 1$ as graph is translated up 1

b) $-1 = k a^0 + 1$

$$k = -2$$

c) $\frac{1}{3} = -2a^{-1} + 1$

$$-\frac{2}{3} = -\frac{2}{a} \implies a = 3$$
$$\therefore y = -2 \times 3^{-x} + 1$$

d) when $y = 0$

$$2 \times 3^{-x} = 1$$
$$3^{-x} = \frac{1}{2}$$
$$3^x = 2$$
$$\therefore x = \frac{\log 2}{\log 3} = \log_3 2$$

P4

$\left(2^{\frac{1}{2}} 3^{\frac{1}{3}}\right)^6$ and $\left(3^{\frac{1}{2}} 2^{\frac{1}{3}}\right)^6$

$= 2^3 \times 3^2$ and $3^3 \times 2^2$

$= 8 \times 9 \quad 27 \times 4$

as $27 \times 4 > 8 \times 9$

$$\therefore \sqrt{3}\sqrt[3]{2} > \sqrt{2}\sqrt[3]{3}$$

P5

$$\lim_{x \to \infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right) \times \frac{e^x}{e^x}$$
$$= \lim_{x \to \infty} \left(\frac{e^{2x} - 1}{e^{2x} + 1}\right) = 1$$

P6 Important to note immediately that domain is $x > 0$

Now $y = \ln x + \ln \frac{1}{x} = \ln 1$
$= 0$

$y = 0$ for $x > 0$

P7

c) $y' = -xe^{-x} + e^{-x}$
$= e^{-x}(1-x) = 0$ for S.P.
$\therefore x = 1$ and then $y = e^{-1}$

d) From the sketch it can be seen that $y = xe^{-x}$ and $y = k$ will have one solution only when $y = k = e^{-1} = \frac{1}{e}$ and $k \leq 0$

P8 $\log \frac{1}{5} < 0$ hence when dividing inequality by a negative we need to change the sign of the inequality

8F

Q1 a) $\$25,000(0.9) = \$22500$

b) $\$25,000(0.9)^2 = \$20250$

c) $\$25,000(0.9)^n$

d) Value after 5 years $= \$25000(0.9)^5$

$= \$14762.25$

$\frac{14762.25}{25000} \times \frac{100}{1} = 59\%$

e) $25000(0.9)^n < 10,000$

$n \log 0.9 < \frac{10}{25}$

$n > 8.7$

9th year

Q2 a) $180(1.2)^{10} = 1114$ students

b) $180(1.2)^n > 1000$

$n \log 1.2 > 5.\dot{5}$

$n > 9.4$

10 years later ie in 2010

Q3 a) $t=0 \quad P=1000 \Rightarrow 1000 = A$

b) $t=2 \quad P=1500 \Rightarrow 1500 = 1000e^{2k}$

$\therefore 2k = \ln \frac{3}{2}$

$k = \frac{1}{2} \ln \frac{3}{2}$

c) $P = 1000e^{4 \times \frac{1}{2} \ln \frac{3}{2}}$

$= 1000e^{\ln(\frac{3}{2})^2} = 1000 \times \frac{9}{4}$

$= 2250$

d) $1000e^{kt} > 1000000$

$\frac{t}{2} \ln \frac{3}{2} > \ln 1000$

$t > 34$ after 34 weeks

Q4 a) $t=0 \quad A=500$

$\therefore 500 = A_0 e^0$

$A_0 = 500$

b) $t=1 \quad A=300$

$\therefore 300 = 500e^{-k}$

$e^{-k} = \frac{3}{5}$

$k = -\ln \frac{3}{5} = \ln \frac{5}{3}$

c) $A = 300\, e^{-10 \ln \frac{5}{3}}$

Note $-10 \ln \frac{5}{3} = \ln\left(\frac{5}{3}\right)^{-10}$

$= \ln\left(\frac{3}{5}\right)^{10}$

and $e^{\ln\left(\frac{3}{5}\right)^{10}} = \left(\frac{3}{5}\right)^{10}$

$\therefore\ A = 300 \times \left(\frac{3}{5}\right)^{10}$

$= 1.8$ so 1 bird

d) $300\left(\frac{3}{5}\right)^{11} = 1.088$

$300\left(\frac{3}{5}\right)^{12} = 0.65$

Q5 a) To show population is declining show $\frac{dP}{dt} < 0$

$\frac{dP}{dt} = -10 \times 0.05\, e^{0.05t}$

$= -\frac{1}{2} e^{0.05t} < 0$ for all t

b) when $t=0$ $P = 200 - 10 = 190$

$95 = 200 - 10\, e^{0.05t}$

$e^{0.05t} = 10.5$

$t = 47$ years.

c) Will $P = 0$?

$10\, e^{0.05t} = 200$

$t = 59.9$ ie 60 years

Yes extinct after 60 years

Q6 a) $P = 2^4 \times 1000 = 16000$

b) $P = 2^{2t} \times 1000$

c) $2^{2t} \times 1000 > 10^6$

$2t \ln 2 > \ln 10^3$

$t > 4.98$

5 hours.

Q7 a) 0.2 mm

b) $1.6 = 0.05 \times 2^n$

$32 = 2^n$

$n = 5$

Q8 a) $f(x) + f(-x) = \frac{1}{1+e^{-x}} + \frac{1}{1+e^{x}}$

$= \frac{e^x}{e^x + 1} + \frac{1}{1+e^x}$

$= \frac{e^x + 1}{e^x + 1} = 1$

b) for an odd function

$f(-x) = -f(x)$

for given function $f(-x) = 1 - f(x)$

To be odd translation down by $\frac{1}{2}$

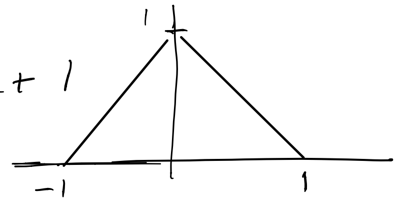
Q9 a) $y' = -e^{-x} = -1$ at $x = 0$

$y - 1 = -(x - 0) \Rightarrow y = -x + 1$

b) Area $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2} u^2$

c) Eq of normal $y = x + 1$

Area $= 1 u^2$



Q10 $P = Ae^{kt}$ when $t = 2$ $P = 100$

$t = 5$ $P = 200$

$100 = Ae^{2k}$

$200 = Ae^{5k}$

$e^{3k} = 2$ dividing

$3k = \ln 2 \Rightarrow k = \frac{\ln 2}{3}$

$100 = A e^{\frac{2}{3}\ln 2} = A e^{\ln(2)^{\frac{2}{3}}}$

$= A \times 2^{2/3}$

$\therefore A = 62.996 = 63$

Q11 a) $3 \times 2^2 \times 2^{3x} = 12 \times (2^3)^x$

$= 12 \times 8^x$

b) $y = k \times (a^b)^x \times a^c$

$= k \times a^c \times (a^b)^x$

$\therefore h = k \times a^c$ $A = a^b$

Q12 a) $\left.\begin{array}{l} 4 = ka \\ 32 = ka^4 \end{array}\right\}$ dividing $8 = a^3$

$\therefore a = 2$ $k = 2$

$y = 2 \times 2^x$

b) $\left.\begin{array}{l} 2 = ka \\ 16 = ka^4 \end{array}\right\}$ $8 = a^3$

$a = 2$ $k = 1$

$\therefore y = 2^x$

c) $\left.\begin{array}{l} 2 = ka^0 \\ 18 = ka^2 \end{array}\right\}$ $k = 2$

$a^2 = 9$ $a = 3$ as $a > 0$

$y = 2 \times 3^x$

d) $20 = ka$

$0.8 = ka^{-1} = \frac{k}{a}$

$25 = a^2 \quad a = 5 \quad k = 4$

$y = 4 \times 5^x$

Q13 a) $x \in \mathbb{R}$ except $e^x \neq 1$

ie $x \neq 0$

b) $e^x + 1 > 0$ for all x

$\therefore x \in \mathbb{R}$

c) $x - 1 > 0 \quad x > 1$

d) $2 - x > 0 \quad x < 2$

e) $3x - 6 > 0 \quad x > 2$

f) $8 - 2x > 0 \quad x < 4$

Q14 a) $x^2 > 0$ for all x so $x \in \mathbb{R}, x \neq 0$

Q15 $M = \frac{M_0}{2}$

$\therefore \frac{M_0}{2} = M_0 e^{-kt}$

$e^{-kt} = \frac{1}{2}$

$-kt = \ln \frac{1}{2}$

$kt = \ln 2$

$t = \frac{1}{k} \ln 2$

Q16 when $t = 0 \quad M = 100 \Rightarrow M_0 = 100$

$M = 100\left(\frac{1}{2}\right)^{\frac{20}{68.9}}$

$= 81.77$

$\doteqdot 82$ kg.

$10 = 100\left(\frac{1}{2}\right)^{\frac{t}{68.9}}$

$\frac{t}{68.9} = \frac{\ln 0.1}{\ln 0.5}$

$t = 228.9$

$\doteqdot 229$ years.

P1 a) $y = e^{x}$

$= e^{x_0}$ at $x = x_0$

b) $y - y_0 = e^{x_0}(x - x_0)$

$y = e^{x_0}(x - x_0) + e^{x_0}$ Note (x_0, y_0) is a point on the curve so it satisfies the equation.

Since line passes through $(0,0)$

$0 = e^{x_0}(-x_0) + e^{x_0}$

$0 = e^{x_0}(1 - x_0)$

$\therefore$ $x_0 = 1$ $y = e$

$P(1, e)$

P2

$f(x) = \dfrac{1 - e^{-\frac{1}{x}}}{1 + e^{-\frac{1}{x}}} \times \dfrac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}}$

$= \dfrac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$

$= -\left(\dfrac{1 - e^{\frac{1}{x}}}{1 + e^{\frac{1}{x}}}\right)$

$= -f(x)$ $\therefore$ odd function

P3 a) We need to find what happens to P as $t \to \infty$ $e^{-kt} \to 0$

$\therefore P \to \frac{a}{b}$.

b) when $t = 0$ $P_0 = \dfrac{a}{b+1}$

$\therefore$ LHS $= \dfrac{1}{P_0} = \dfrac{b+1}{a}$

$= \dfrac{b}{a} + \dfrac{1}{a}$

$= \dfrac{1}{\frac{a}{b}} + \dfrac{1}{a} = \dfrac{1}{c} + \dfrac{1}{a}$

c) $P(t) = a(b + e^{-kt})^{-1}$

$P'(t) = \dfrac{ak\,e^{-kt}}{(b + e^{-kt})^2}$

as $a > 0$ $k > 0$ and $e^{-kt} > 0$ for all t

$P'(t) > 0$

$\therefore$ always increasing

d) $t = 0$ $P = 800$ and as $t \to \infty$ $P \to 4000$

i.e. $C = \frac{a}{b} = 4000$

$\dfrac{1}{800} = \dfrac{1}{a} + \dfrac{1}{4000}$

$\dfrac{1}{1000} = \dfrac{1}{a}$ $\Rightarrow$ $a = 1000$ $b = \frac{1}{4}$

e) No

P4 a) $y = a^x$ and $y = \log_a x$ are inverse functions and the graphs are reflections about the line $y = x$. As a gets smaller the curves will touch on $y = x$
If point of contact is $x = \beta$ then $y = \beta$ since on line $y = x$ and on curve.

$y = a^x \qquad y = a^\beta$

$\Rightarrow \qquad a^\beta = \beta$

b) $\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$

$= \ln a \, e^{x \ln a}$

$= a^x \ln a$

c) Hence gradient of tangent $y = a^x$ at $x = \beta$

$= a^\beta \ln a$

However tangent is line $y = x$ whose gradient is $= 1$

$\therefore a^\beta \ln a = 1$

d) $a^\beta = \beta \qquad \therefore \beta \ln a = 1$

So $\qquad \beta = \frac{1}{\ln a}$

e) Consider curve $y = \log_a x$
We know (β, β) is a pt on it

$\therefore \beta = \log_a \beta$

or We proved $a^\beta = \beta$ in a)
hence $\log_a \beta = \beta$

f) as $\beta = \log_a \beta$ and $\beta = \frac{1}{\ln a}$

$\log_a \beta = \frac{1}{\ln a}$

$\frac{\ln \beta}{\ln a} = \frac{1}{\ln a}$

$\therefore \ln \beta = 1$

$\beta = e$

g) $\therefore a^e = e$

$a = \sqrt[e]{e}$

REVIEW

Q1 a) $\frac{\log 2^5}{\log 3} \times \frac{2\log 3}{\log 2} = 10$

b) $\log_9 (3\log_2 2) = \log_9 3$
$= \log_9 9^{1/2}$
$= \frac{1}{2}$

c) $\log_4 4^{\frac{1}{2}} = \frac{1}{2}$
$4^{\frac{1}{2}} = -$

Q2 a) $A(0,1)$
$D(1,0)$ $\therefore E(1,1)$
$B(4\ 1)$ $C(4,0)$

b) $3\ u^2$

Q3 a) $2\log_x p = 2 \times 1.6 = 3.2$
b) $\frac{1}{2}\log_x p = \frac{1}{2} \times 1.6 = 0.8$
c) $-\log_x p = -1.6$
d) $x \log_x p = 1.6x$

Q4 $-\log_2 x - 2\log_2 x - 3\log_2 x - 4\log_2 x = -40$
$-10\log_2 x = -40$
$\log_2 x = 4$
$x = 2^4 = 16$

Q5 a) $\log a - 2\log b = x - 2y$
b) $\log b + \frac{1}{2}\log c = y + \frac{1}{2}z$

Q6 $\ln y = 2\ln e - \ln x^2 + \ln x$
$\ln y = \ln\left(\frac{e^2}{x^2} \times x\right)$
$y = \frac{e^2}{x}$

Q7 $\ln[x(x+2)] = 1$
$x^2 + 2x - e = 0$
$x = \frac{-2 \pm \sqrt{4+4e}}{2} = -1 + \sqrt{1+e}$

$= 0.93$ since $x > 0$

Q8 a) $4 - 3x > 0$
$x < \frac{4}{3}$

b) All real x

Q9 a) $\log_6(3\times4\times4) = a + 2b$

b) $\log_6(4\times2) = \frac{3}{2}b$

c) $\log_6(8\times3) = a + \frac{3}{2}b$ or $\log_6(6\times4) = 1 + b$

d) $\log_6 2 - \frac{1}{2}\log 3 = \frac{1}{2}b - \frac{1}{2}a$

Q10

Q11 $A(1,0)$ $D(0,2)$ $C(e^2, 2)$ $B(e^2, 0)$

Q12 a) $3^{2x}\times3^2 - 3^x\times3^3 - 3^x + 3$

$= 9\times3^x(3^x - 3) - (3^x - 3) = (9\times3^x - 1)(3^x - 3)$

b) $3^x = \frac{1}{9}$ or $3^x = 3$

$x = -2$ or $x = 1$

Q13 Let $\ln x = u$

$u^2 - 3u - 4 = 0$

$(u-4)(u+1) = 0$

$u = 4$ $u = -1$

$x = e^4$ $x = e^{-1} = \frac{1}{e}$

Q14 $e^{2x} - 2e^x + 1 = 0$

$(e^x - 1)^2 = 0$

$e^x = 1$ $x = 0$

Q15 $\log_{\frac{1}{b}} a = \dfrac{\log_b a}{\log_b \frac{1}{b}} = \dfrac{\log_b a}{-\log_b b}$

$= -\log_b a$

Q16 a) Since $(0,-1)$ lies on curve $-1 = k + c$

$(-1, -\frac{1}{2})$ " " $-\frac{1}{2} = ka^{-1} + c$

assymptote $y = -2 \Rightarrow c = -2$

$\therefore k = 1$

$-\frac{1}{2} = \frac{1}{a} - 2$

$a = \frac{2}{3}$

$y = \left(\frac{2}{3}\right)^x - 2$

b) when $x = -3$ $y = \frac{27}{8} - 2 = \frac{11}{8}$

Q17 a) $(x-3)^2 = 49$

$x - 3 = \pm 7$ $\quad x = 10 \quad$ $x=-4$ is not a valid soln.

b) $x+3 = (x-3)^2$

$x+3 = x^2 - 6x + 9$

$x^2 - 7x + 6 = 0$

$(x-6)(x-1) = 0$

$x = 6$ or $x = 1$

Only $x=6$ is a valid soln

c) $(2x-1)(2x+1) = 3$

$4x^2 - 1 = 3$

$4x^2 = 4$

$x = 1$

d) $(\ln x)^2 - 7\ln x + 12 = 0$

$(\ln x - 3)(\ln x - 4) = 0$

$\ln x = 3$ or $\ln x = 4$

$x = e^3$ or $x = e^4$

Q18

$$\text{RHS} = \log_{ab} x = \frac{\log_x x}{\log_x ab}$$

$$= \frac{1}{\log_x a + \log_x b}$$

$$= \frac{1}{\frac{\log_a a}{\log_a x} + \frac{\log_b b}{\log_b x}}$$

$$= \frac{1}{\frac{1}{\log_a x} + \frac{1}{\log_b x}} = \text{LHS}$$