1

FURTHER FUNCTIONS

Exercise 1A

Reciprocal and square root graphs

Fundamentals

Fundamentals 1

Let $P(a, b)$ be a point on $y = f(x)$.

(a) The image of P under the transformation $y = \dfrac{1}{f(x)}$ is ____

(b) The image of P under the transformation $y = \sqrt{f(x)}$ is ____

Fundamentals 2

(a) As $f(x)$ increases, the graph of $\dfrac{1}{f(x)}$ increases/decreases (circle one).

(b) As $f(x)$ decreases, the graph of $\dfrac{1}{f(x)}$ increases/decreases (circle one).

(c) As $f(x) \to 0^+$, the graph of $\dfrac{1}{f(x)} \to$ ____.

(d) As $f(x) \to 0^-$, the graph of $\dfrac{1}{f(x)} \to$ ____.

(e) As $f(x) \to \infty$, the graph of $\dfrac{1}{f(x)} \to$ ____.

(f) As $f(x) \to -\infty$, the graph of $\dfrac{1}{f(x)} \to$ ____.

(g) All x-intercepts from $y = f(x)$ become v____ a______ on $y = \dfrac{1}{f(x)}$.

Fundamentals 3

Consider the equation of $y = \sqrt{f(x)}$.

(a) What happens if $f(x) < 0$?

(b) If $0 < f(x) < 1$, then $y = \sqrt{f(x)}$ is higher/lower (circle one) than $y = f(x)$.

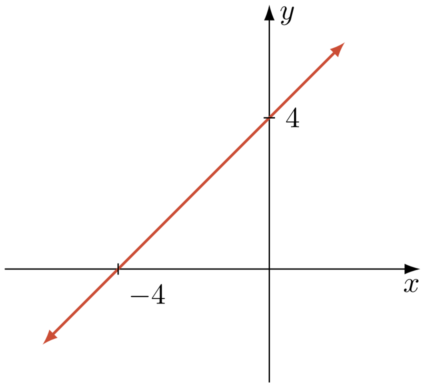
(c) If $f(x) > 1$, then $y = \sqrt{f(x)}$ is higher/lower (circle one) than $y = f(x)$.

(d) If $f(x)$ has a zero at $x = \alpha$, then $y = \sqrt{f(x)}$ also has a zero at $x = \alpha$. However, there will be a v______ tangent at $x = \alpha$, provided that $f'(\alpha) \neq 0$.

Fundamentals 4

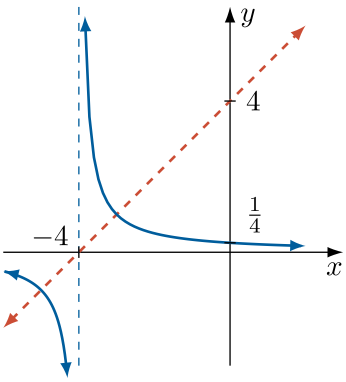
Explain the difference between the graphs of $y = \sqrt{f(x)}$ and $y^2 = f(x)$.

Question 1 The diagram below shows the graph of $y = f(x)$.

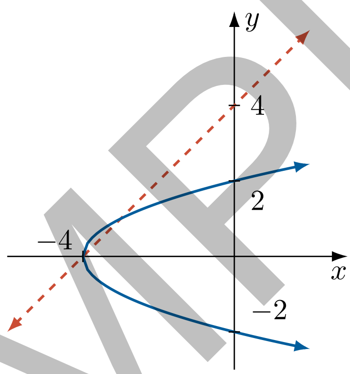


The following diagrams show the graphs of $y = \sqrt{f(x)}$, $y = \dfrac{1}{f(x)}$ and $y^2 = f(x)$ in a random order. Write down the transformation that matches each of the diagrams.

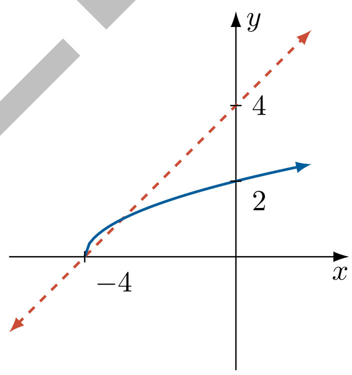
(a)



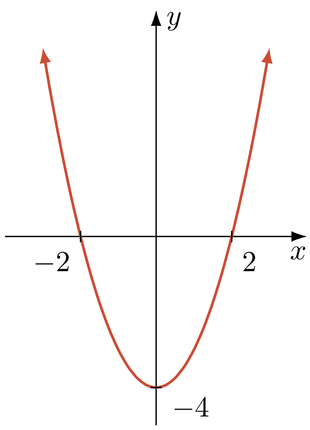
(b)



(c)

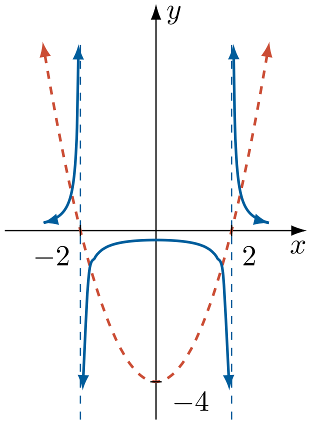


Question 2 The diagram below shows the graph of $y = f(x)$.

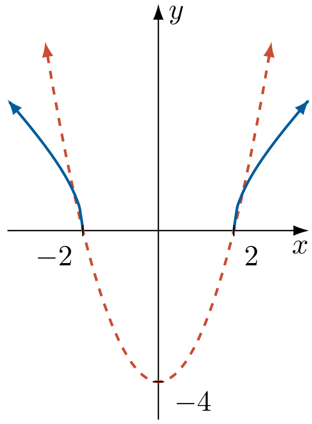


The following diagrams show the graphs of $y = \sqrt{f(x)}$, $y = \dfrac{1}{f(x)}$ and $y^2 = f(x)$ in a random order. Write down the transformation that matches each of the diagrams.

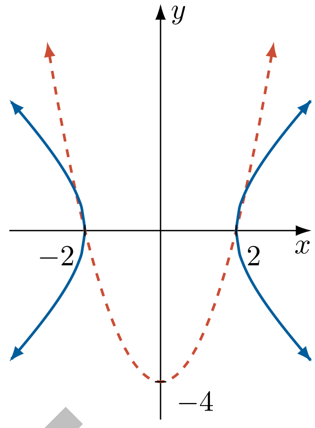
(a)



(b)



(c)



Question 3

(a) Draw the graph of $y = x^2 - 2x$, labelling all important features.

(b) Hence, draw a sketch of $y = \sqrt{x^2 - 2x}$ on the same set of axes.

Question 4 Use a similar technique to sketch the graph of the following.

(a) $y = \sqrt{2x - 1}$ (b) $y = \dfrac{1}{\sqrt{x}}$ (c) $y = \sqrt{x^3 - 4x}$

(d) $y = \sqrt{x^3 + 1}$ (e) $y = \sqrt{2 + x - x^2}$ (f) $y = \sqrt{4 + 2^{-x}}$

Question 5 By first drawing $y = f(x)$, sketch the following graphs of $y^2 = f(x)$.

(a) $y^2 = x + 2$ (b) $y^2 = x^2 - 4$ (c) $y^2 = 2^x - 1$

Question 6

(a) Draw the graph of $y = 4x - x^2$, labelling all important features.

(b) Hence, draw a sketch of $y = \dfrac{1}{4x - x^2}$ on the same set of axes.

Question 7 Use a similar technique to sketch the graph of the following. Draw the 'original' graph as a dashed curve, and draw the final answer on the same set of axes.

(a) $y = \dfrac{1}{2x - 1}$ (b) $y = \dfrac{1}{\sqrt{x}}$ (c) $y = \dfrac{1}{x^2}$

(d) $y = \dfrac{1}{x^2 + x - 2}$ (e) $y = \dfrac{1}{x^3 + 1}$ (f) $y = \dfrac{1}{1 + 2^x}$

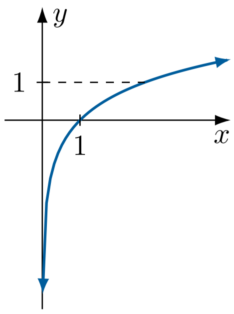
Question 8

(a) Sketch the graph of $y = x^3$.

(b) Hence, sketch the graph of the following.

(i) $y = \sqrt{x^3}$ (ii) $y^2 = x^3$ (iii) $y = \dfrac{1}{x^3}$

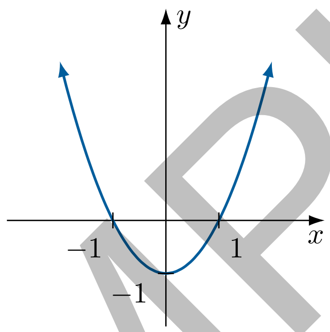
Question 9 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

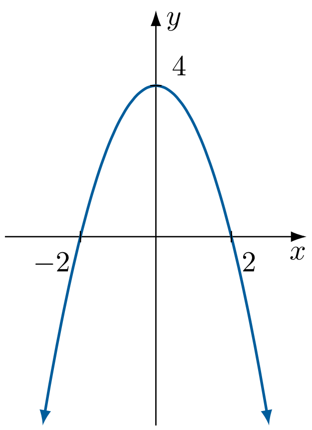
Question 10 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

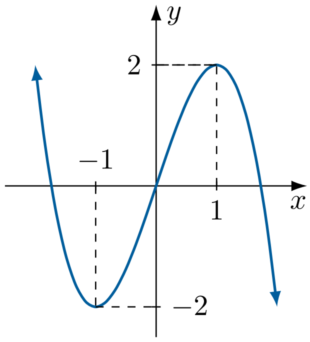
Question 11 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

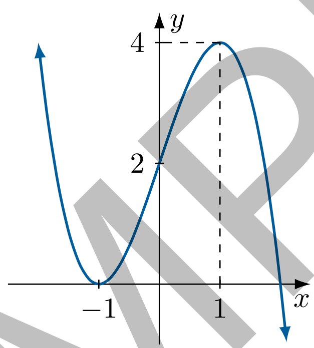
Question 12 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

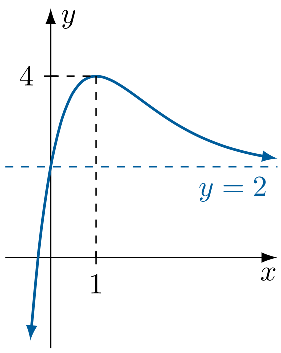
Question 13 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

Question 14 The diagram below shows the graph of $y = f(x)$.



On separate axes, sketch the graph of

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \sqrt{f(x)}$ (c) $y^2 = f(x)$

Challenge Problems

Problem 1 Sketch the following.

(a) $y = \dfrac{1}{\sqrt{x^2 - 1}}$

(b) $y = \dfrac{1}{\sqrt{1 - x^2}}$

Problem 2 Sketch the graph of $y = \dfrac{2^x}{1 + 2^x}$.

Hint: Divide the top and bottom by 2^x

Problem 3 [Kampyle of Eudoxus]

(a) Sketch the graph of $y = x^4 - x^2$.

(b) Hence, sketch the graph of $y^2 = x^4 - x^2$, which is called the *Kampyle of Eudoxus* named after the ancient Greek astronomer and mathematician Eudoxus of Cnidus (408 BC – 347 BC).

(c) Use graphing software to sketch $y^2 = x^4 - x^2$ and $y = x^2$ on the same set of axes. State what you observe.

(d) Prove your observation.

Problem 4 [Lemniscate]

Sketch the graph of $y^2 = x^2 - x^4$.

Problem 5 [Calculus required]

Prove the following statements about the reciprocal and square root graphs.

(a) If $f(x)$ has a root at $x = \alpha$, and $f'(\alpha) \neq 0$, then $y = \sqrt{f(\alpha)}$ has a vertical tangent at $x = \alpha$.

(b) If $f(x)$ has a stationary point at $x = \alpha$, then $y = \dfrac{1}{f(x)}$ also has a stationary point at $x = \alpha$.

(c) If $f(x)$ has a turning point at $x = \alpha$, then $y = \dfrac{1}{f(x)}$ also has a turning point at $x = \alpha$, but with opposite concavity.

Exercise 1G

Parametric forms

Fundamentals

Fundamentals 1

(a) A C________ equation is an equation relating two variables x and y.

(b) These variables can be expressed as functions of a third variable called a p________.

(c) This p________ can be used to study the ___ or ___-coordinates individually, rather than studying them together all the time.

(d) Every point on the curve is now defined by only o___ number, which is the value of the p________.

(e) For a given Cartesian equation, the parametrisation is/is not (circle one) unique. In other words, a given Cartesian equation may/may not (circle one) have many parametric equations to represent it.

Fundamentals 2

(a) To obtain the C________ equation from the parametric equation, we need to e________ the parameter.

(b) This can often be done for most problems either by making the parameter the subject from one equation first and then s________ into the other, or by using a t________ identity.

Fundamentals 3

(a) The usual parametrisation for the circle $x^2 + y^2 =$ ___ is $x = r\cos\theta$ and $y =$ ________.

(b) It relies on the trigonometric identity ________________.

(c) If the circle is centred at (a, b), then a parametrisation is $x = a +$ ________ and $y =$ ___ $+ r\sin\theta$.

Question 1 Consider the curve defined parametrically by $x = t - 1$ and $y = t + 1$

(a) Complete the following table.

t	-2	-1	0	1	2
x					
y					

(b) Eliminate the parameter and hence find the Cartesian equation.

(c) What value of t yields the coordinate $(4, 6)$?

(d) Sketch the graph and plot the points corresponding to $t = 0$, 1, 2 on it.

Question 2 Consider the curve defined parametrically by $x = 3t$ and $y = t^2$

(a) Complete the following table.

t	-2	-1	0	1	2
x					
y					

(b) Eliminate the parameter and hence find the Cartesian equation.

(c) What value of t yields the coordinate $(6, 4)$?

(d) Sketch the graph and plot the points corresponding to $t = 0,\ 1,\ 2$ on it.

(e) Let T be the point on the parabola with parameter t. As t varies, the position of T will also vary. Describe what happens to T as $t \to \pm\infty$.

Question 3 For each of the following, eliminate the parameter and hence state the Cartesian equation.

(a) $x = 2t$, $y = 3t$

(b) $x = 3 + t$, $y = 2t$

(c) $x = 2 - 3t$, $y = 4 + 2t$

(d) $x = 4t$, $y = 16t^2$

(e) $x = 3t$, $y = 6t^2$

(f) $x = t - 3$, $y = 1 - t^2$

Question 4 For each of the following, show that the Cartesian equation is a circle and state the centre and radius.

(a) $x = \cos\theta$, $y = \sin\theta$

(b) $x = 2\cos\theta$, $y = 2\sin\theta$

(c) $x = -1 + \cos\theta$, $y = 2 - \sin\theta$

(d) $x = 4 + 3\cos\theta$, $y = -5 + 3\sin\theta$

Question 5 For each of the following circles, write down a suitable parametric equation.

(a) $x^2 + y^2 = 16$

(b) $(x-2)^2 + (y+5)^2 = 9$

(c) $x^2 + 6x + y^2 - 2y - 15 = 0$

Question 6 Sketch the following parametrically defined curves.

(a) $x = 3t - 5$, $y = 2t + 1$

(b) $x = 2t$, $y = t^2 - 1$

(c) $x = t - 2$, $y = 2t^2 + 1$

Question 7 [Trick question]
Find the Cartesian equation of the following.

(a) $x = 2t + 3$, $y = 5$

(b) $x = -2$, $y = t^3 + 1$

Question 8 [Importance of checking domain and range]

(a) Find the Cartesian equation of $(t^2, t^2 - 1)$.

(b) Bob claims that the graph is just the graph of $y = x - 1$ whereas Mary claims that it is only the right-hand side of the graph. By substituting a few values of t and plotting the resultant point, determine who is correct.

(c) Explain why they are not the same.

Question 9 Use a similar technique to Question 8 to find and sketch the Cartesian equation of the following. Remember to state any restrictions where necessary.

(a) $(3 - t^2,\ 2 + t^2)$ (b) $(\sqrt{t-1},\ t)$ (c) $(2,\ t^2 + 1)$

Challenge Problems

Problem 1 For the following Cartesian equations, find two possible parametric representations.

(a) $y = 2x + 3$ (b) $y = 4x^2 + 1$ (c) $x^2 + y^2 = 9$

Problem 2 [Folium of Descartes]
Consider the curve defined parametrically by

$$x = \frac{3t}{1+t^3}$$
$$y = \frac{3t^2}{1+t^3}$$

(a) Show that $\dfrac{y}{x} = t$.

(b) Deduce that $x^3 + y^3 = 3xy$.

(c) Use graphing software to produce a sketch of the *Folium of Descartes*.

Problem 3 [Parametrisation of the ellipse]

The ellipse has equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where a and b are constants. Find a suitable parametrisation for the ellipse by modifying the standard parametrisation for the circle.

Problem 4 [More advanced algebraic parametrisations]
Eliminate the parameter in each of the following.

(a) $x = t + \frac{1}{t}$
$y = t^2 + \frac{1}{t^2}$

(b) $x = t + \frac{1}{t}$
$y = t - \frac{1}{t}$

Problem 5 [More advanced trigonometric parametrisations]
Eliminate the parameter in each of the following.

(a) $x = \sec\theta$
$y = \tan\theta$

(b) $x = \cos\theta + \sin\theta$
$y = \cos\theta - \sin\theta$

Chapter 1 Review

Further Functions

Review

Question 1 By first drawing a graph of $y = f(x)$, sketch a graph of $y = \dfrac{1}{f(x)}$.

(a) $f(x) = x + 1$

(b) $f(x) = x^2 + 2$

(c) $f(x) = x^2$

(d) $f(x) = x^2 - 4$

Question 2 By first drawing a graph of $y = f(x)$, sketch a graph of $y^2 = f(x)$.

(a) $f(x) = 2x - 4$

(b) $f(x) = x^2 + 1$

(c) $f(x) = x^2 - 16$

(d) $f(x) = 16 - x^2$

Question 3 By first drawing a graph of $y = f(x)$, sketch a graph of $y = |f(x)|$.

(a) $f(x) = 3x + 4$

(b) $f(x) = x^2 - 16$

(c) $f(x) = (x - 1)(x^2 - 4)$

(d) $f(x) = \sqrt{x} - 1$

Question 4 By first drawing a graph of $y = f(x)$, sketch a graph of $y = f(|x|)$.

(a) $f(x) = 6 - 2x$

(b) $f(x) = x^2 - 2x - 8$

(c) $f(x) = x^3 - 9x$

(d) $f(x) = \sqrt{x + 1}$

Question 5 By first drawing a graph of $y = f(x)$ and $y = g(x)$, sketch a graph of $y = f(x) + g(x)$.

(a) $f(x) = x,\ g(x) = -\sqrt{x}$

(b) $f(x) = x,\ g(x) = \sqrt{1 - x^2}$

(c) $f(x) = x^2,\ g(x) = \dfrac{1}{x}$

(d) $f(x) = \sqrt{x},\ g(x) = \dfrac{1}{x}$

Question 6 By first drawing a graph of $y = f(x)$ and $y = g(x)$, sketch a graph of $y = f(x)g(x)$.

(a) $f(x) = x,\ g(x) = x^2 + 1$

(b) $f(x) = x,\ g(x) = \sqrt{1 - x^2}$

(c) $f(x) = x^2,\ g(x) = \sqrt{1 - x^2}$

(d) $f(x) = x^2,\ g(x) = 4^{-x}$

Question 7

(a) Sketch the graph of $y = \sqrt{x} - 1$.

(b) Hence, sketch the graph of $y = \dfrac{1}{\sqrt{x} - 1}$.

Question 8 Solve the following inequalities.

(a) $x^2 \geq 25$

(b) $4x > x^2$

(c) $x^2 - x - 20 \leq 0$

(d) $12 - x - x^2 < 0$

Question 9 Solve the following inequalities.

(a) $\dfrac{2}{x-1} \geq 3$

(b) $\dfrac{3}{2x+1} \leq -1$

(c) $\dfrac{x}{x+1} \geq 2$

(d) $\dfrac{x-1}{x+1} \leq 4$

Question 10 Solve the following inequalities.

(a) $|x-2| \geq 5$

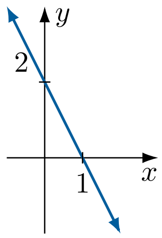
(b) $|2x+3| \leq 9$

(c) $|3-2x| > 7$

(d) $\left|\dfrac{3x+1}{2}\right| > 5$

Question 11 For each of the following graphs, sketch the inverse function.

(a)



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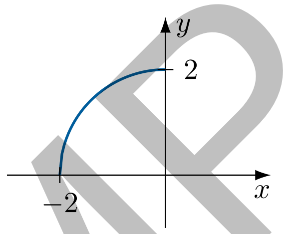

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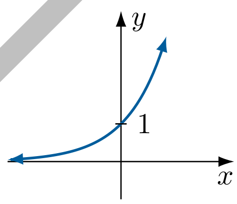

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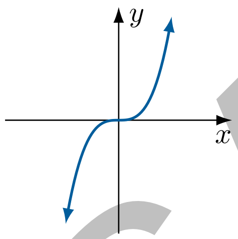

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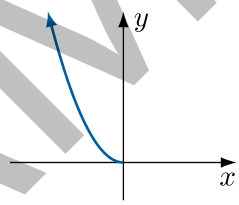

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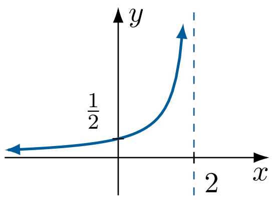

Question 12 For the following functions, find the equation of $f^{-1}(x)$ and hence show that

$$f^{-1}\left(f(x)\right) = f\left(f^{-1}(x)\right) = x$$

(a) $f(x) = 5 - x$

(b) $f(x) = 3x - 1$

(c) $f(x) = \sqrt{x}$

(d) $f(x) = \dfrac{1}{x-1}$

Question 13 Let $f(x) = x^2 - 8x$.

(a) Let $x \in [p, \infty)$ be a domain so that $f^{-1}(x)$ exists. Find the smallest value of p.

(b) Find the equation of the inverse and hence sketch the graph of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 14 Find the inverse of the following. If required, restrict the domain to contain only positive values of x.

(a) $y = x^2 - 6x + 14$

(b) $y = x^4 - 1$

(c) $y = \dfrac{2}{x^2}$

(d) $y = \sqrt{9 - x^2}$

Question 15

(a) Find the domain and range of $f(x) = \dfrac{3}{x-2}$.

(b) Find the equation of $f^{-1}(x)$.

(c) What are the x-coordinates of where the graphs of $y = f(x)$ and $y = f^{-1}(x)$ intersect?

(d) Sketch $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes.

Question 16 Eliminate the parameter t and hence find the Cartesian equation of the following.

(a) $x = 3t - 1$
$y = 2t + 3$

(b) $x = 2t$
$y = t^2 - 1$

(c) $x = 2at$
$y = at^2$

(d) $x = 2t - 1$
$y = t^2 - t$

(e) $x = 2\cos\theta$
$y = 2\sin\theta$

(f) $x = 2 + 2\cos\theta$
$y = -3 + 2\sin\theta$

Question 17 Write down the centre and radius of the circles defined parametrically by the following.

(a) $x = -2 + 5\cos\theta$
$y = 3 + 5\sin\theta$

(b) $x = 4 - 3\cos\theta$
$y = -1 + 3\sin\theta$

Question 18 Express the quadratic function $y = x^2 + 2x - 1$ in parametric form, given that $x = 2t - 1$.

Question 19 Show that the point $P\left(ap, \dfrac{a}{p}\right)$ is an appropriate parametrisation of $xy = a^2$.

Question 20 Draw the graph of the following by addition of ordinates.

(a) $y = |x| + |x - 2|$

(b) $y = |x| - |x - 2|$

Investigation Task

Further Reflections

So far, you have learned the following transformations (and their combinations), which require reflections.

$$y = -f(x), \quad y = |f(x)|$$
$$y = f(-x), \quad y = f(|x|)$$

This investigation task will take further the study of reflections.

Question 1 Create a function of your choice that lies both above and below the x-axis, and call it $f(x)$. Construct a graph of it using graphing software.

(a) Use graphing software to sketch $|y| = f(x)$ on the same set of axes as $y = f(x)$. Comment on your findings.

(b) Write down a set of instructions for a student on how to draw $|y| = f(x)$ for any given function.

(c) Bob makes the following argument.

> "Much like how $|x| = 5$ implies $x = \pm 5$, we can say that $|y| = f(x)$ implies $y = \pm f(x)$. So, the graph of $|y| = f(x)$ is just the positive and negative graphs on the same set of axes."

Is Bob's answer correct? If not, then is it partially correct or not-at-all correct? Give a detailed response and provide examples or counter-examples where necessary.

(d) Suppose $f(x) = -x^2 - 1$. Draw the graph of $|y| = f(x)$ and comment on your findings with justification. Repeat this for $f(x) = x^2 + 1$ and similarly comment on your findings.

Question 2 Create a function of your choice and call it $f(x)$. Construct a graph of it using graphing software.

(a) Use graphing software to sketch $y = f(4 - x)$ on the same set of axes as $y = f(x)$. Comment on your findings.

(b) Write down a detailed set of instructions for a student on how to draw $y = f(a - x)$ for any given function and for various values of a.

(c) Explain why the graph of $y = f(a - x)$ has the effect that it does on the graph of $f(x)$.